Generalized Iteration and Coiteration for Higher-Order Nested Datatypes

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Abstract. We solve the problem of defining generalized iteration (generalized in the direction of generalized folds by Bird and Paterson) and its dual for inductive and coinductive constructors (nested datatypes) of *arbitrary* rank by appropriately generalizing Mendler-style (co)iteration. Characteristically to Mendler-style schemes of disciplined (co)recursion, the schemes we propose do not rest on notions like positivity or monotonicity of a constructor and facilitate programming in a natural and elegant style close to programming with the customary `letrec` construct, where the typings of the schemes, however, guarantee termination. For rank 2, a smoothened version of Bird and Paterson's generalized folds and its dual are achieved, for rank 1, the schemes instantiate to Mendler's original (re)formulation of iteration and coiteration. Several examples demonstrate the power of the approach. Strong normalization of our proposed extension of system F^ω of higher-order parametric polymorphism is proven by a reduction-preserving embedding into pure F^ω.

1 Introduction

Bird and Paterson [7] and colleagues have studied the problem of identifying workable schemes for defining functions for nested or heterogeneous datatypes, i.e., inductive and coinductive constructors of rank 2 (type transformers), and put forth a scheme called *generalized folds* enabling natural definitions of functions such as substitution for variables in de Bruijn notation.

In [2], two of the authors of the present paper showed that, making good use of right notions of containment and monotonicity of constructors, the schemes of *iteration* and *coiteration* are extensible to monotone (co)inductive constructors of

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any finite kind. The project of the present paper was to similarly extend the more liberal generalized folds for all finite kinds. This has been achieved here thanks to two ideas: a simple, but powerful generalization of the notion of constructor containment, and reformulation of the schemes in the style originated by Mendler [14]. The result is a concise extension of system F^ω of higher-order parametric polymorphism with (co)inductive constructors of any finite kind, equipped with Mendler-style generalized (co)iteration. Switching to Mendler style was not intentional, but in the end turned out rewarding. The reasons are the following.

Firstly, any syntactic positivity requirement can be avoided in the formation rules of (co)inductive types. This is beneficial as positivity and map terms associating to positive constructors would have to be defined by induction outside the system; parametrically polymorphic quantification over all positive constructors is impossible. Moreover, for higher kinds, there is no obvious canonical definition of positivity, although attempts of definition exist [11]. Replacing positivity with monotonicity [13, 2] gives an improvement, but formulating the systems and especially programming remain a bit clumsy.

Secondly, Mendler style facilitates a programming style very close to programming with general recursion (i.e., the `letrec` construct). The computation rules for Mendler-style disciplined (co)recursion schemes are nearly identical to the rule of `letrec`, the restrictive typings however ensure that all computations terminate.

Thirdly, Mendler-style disciplined (co)recursion schemes tend to be amenable for generalizations whereas conventional ones—making use of map terms or monotonicity witnesses—typically get complicated. Examples are: primitive (co-) recursion [14], course-of-value (co)iteration [17, 18], iteration over multiple inductive types at the same time [18] and—as the examples of this paper testify—generalized iteration in the sense of generalized folds.

The paper is organized as follows. In Sect. 2, we review our starting point system F^ω of higher-order parametric polymorphism. In Sect. 3, we present our system MIt^ω of (co)inductive constructors of finite ranks with generalized Mendler-style iteration and also consider the first programming examples. The discussion of examples is continued in Sect. 4 where a subsystem with a more restricted iteration scheme with the flavour of generalized folds is introduced. The embedding of MIt^ω into F^ω is presented in Sect. 5. We conclude with a summary and discussion of related work.

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2 System F^ω

Our development of higher-order datatypes takes place within a conservative extension of Curry-style system F^ω by nullary and binary sums and products and existential quantification. It contains three syntactic categories:

Kinds. We denote kinds by the letter κ. For *the pure kind of rank n* we introduce the special name κn.

$$\begin{array}{lll} \kappa & ::= * \mid \kappa \to \kappa' \\ \kappa 0 & := & * \\ \kappa(n+1) & := & \kappa n \to \kappa n \end{array}$$

Examples for pure kinds are $\kappa 0 = *$, *types*, $\kappa 1 = * \to *$, *type transformers* and $\kappa 2 = (* \to *) \to * \to *$ *transformers of type transformers*. Note that each kind κ' can be uniquely written as $\boldsymbol{\kappa} \to *$, where we write $\boldsymbol{\kappa}$ for $\kappa_1, \ldots, \kappa_n$ and set $\kappa_1, \ldots, \kappa_n \to \kappa := \kappa_1 \to \ldots \to \kappa_n \to \kappa$, assuming that $\to$ associates to the right. Also set $|\kappa_1, \ldots, \kappa_n| := n$. If $\boldsymbol{\kappa}' = \kappa'_1, \ldots, \kappa'_n$, hence $|\boldsymbol{\kappa}'| = |\boldsymbol{\kappa}|$, then set $\boldsymbol{\kappa} \to \boldsymbol{\kappa}' := \kappa_1 \to \kappa'_1, \ldots, \kappa_n \to \kappa'_n$. This last abbreviation does not raise any ambiguities due to the required $|\boldsymbol{\kappa}'| = |\boldsymbol{\kappa}|$.

Constructors. Uppercase latin letters denote constructors, given by the following grammar.

$$\begin{array}{c} A, B, F, G ::= X \mid \lambda X^\kappa.F \mid F\,G \mid \forall F^\kappa.\, A \mid \exists F^\kappa.\, A \mid A \to B \\ \mid 0 \mid A + B \mid 1 \mid A \times B \end{array}$$

We identify β-equivalent constructors. A constructor F has kind κ if there is a context Γ such that $\Gamma \vdash F : \kappa$. The kinding rules for the constructors can be found in Appendix A. It also contains the typing rules for the terms and the reduction rules.

Preferably we will use letters A, B, C, D for constructors of rank 0 *(types)* and F, G, H for constructors of rank 1. If no kinds are given and cannot be guessed from the context, we assume $A, B, C, D : *$ and $F, G, H : \kappa 1$. Write $\mathsf{Id}_\kappa := \lambda X^\kappa.X$ for the identity constructor. If the kinding is clear from the context, we just write Id. Constructor application associates to the left, i. e., $F\,G_1 \ldots G_n = (\ldots(F\,G_1)\ldots)\,G_n$. Setting $\boldsymbol{G} := G_1, \ldots, G_n$, the constructor $F\,G_1 \ldots G_n$ is also written as $F\boldsymbol{G}$. Sums and products can inductively be extended to all kinds: For $F, G : \kappa_1 \to \kappa_2$ set $F + G := \lambda X^{\kappa_1}.FX + GX$ and $F \times G := \lambda X^{\kappa_1}.FX \times GX$.

Objects (Curry terms). Lower case letters denote terms.

$$\begin{array}{c} r, s, t ::= x \mid \lambda x.t \mid r\,s \mid \mathsf{abort}\,r \mid \mathsf{inl}\,t \mid \mathsf{inr}\,t \mid \mathsf{case}\,(r,\, x.\,s,\, y.\,t) \\ \mid \langle\rangle \mid \langle t_0, t_1\rangle \mid r.0 \mid r.1 \mid \mathsf{pack}\,t \mid \mathsf{open}\,(r,\, x.\,s) \end{array}$$

Most term constructors are standard; "pack" introduces and "open" eliminates existential quantification. The polymorphic identity $\lambda x.x : \forall A.\, A \to A$ will be denoted by id. We write $f \circ g$ for function composition $\lambda x.\, f\,(g\,x)$. Application $r\,s$ associates to the left, hence $r\boldsymbol{s} = (\ldots(r\,s_1)\ \ldots\ s_n)$ for $\boldsymbol{s} = s_1, \ldots, s_n$.

A term t has type A if $\Gamma \vdash t : A$ for some context Γ. The relation $\longrightarrow$ denotes the usual one-step β-reduction which is confluent, type preserving and strongly normalizing. As mentioned above, the exact typing and reduction rules can be found in Appendix A.

In the following we will refer to the here defined system simply as "F^ω".

3 Generalized Mendler-Style Iteration and Coiteration

In this section, we recap and extend the notions of containment and monotonicity presented in [2]. On top of these notions, we define the system MIt^ω of generalized (co)iteration for arbitrary ranks, which we then specialize to rank 1 (types) and rank 2 (type transformers). To give a feel for our system, we spell out some examples involving *nested datatypes* [6].

3.1 Containment and Monotonicity of Constructors

Containment. The key to extending Mendler-style iteration and coiteration [15] to finite kinds consists in identifying an appropriate notion of predicate of containment between two constructors of the same kind κ. For types, the canonical choice is implication. For an arbitrary kind $\kappa = \boldsymbol{\kappa} \to *$, the easiest notion is "pointwise implication": The constructor $\subseteq_\kappa \colon \kappa \to \kappa \to *$ is defined by

$$F \subseteq_\kappa G := \forall \boldsymbol{X}^{\boldsymbol{\kappa}}.\, F\boldsymbol{X} \to G\boldsymbol{X},$$

hence $F \subseteq_\kappa G$ is a type seen to be the proposition stating that F is contained in G. A more refined notion $\leq$ was employed already in [2], where (co)iteration for monotone (co)inductive constructors of higher kinds was studied. In the case of type transformers F, G of kind $\kappa 1$, the definition (suggested by Peter Hancock) was

$$F \leq_{\kappa 1} G := \forall A \forall B.\, (A \to B) \to FA \to GB.$$

In order to be able to extend Mendler (co)iteration to higher kinds so that generalized folds [7] are covered, we have to relativize the notion $\leq_\kappa$, $\kappa = \boldsymbol{\kappa} \to *$, of [2] to a vector $\boldsymbol{H}$ of constructors of kinds $\boldsymbol{\kappa} \to \boldsymbol{\kappa}$. For $\boldsymbol{H}$ a vector of identity constructors, the new notion coincides with the one proposed in loc. cit.

For every kind $\kappa = \boldsymbol{\kappa} \to *$, we define a constructor $\leq_\kappa^{(-)} \colon (\boldsymbol{\kappa} \to \boldsymbol{\kappa}) \to \kappa \to \kappa \to *$ by structural recursion on κ as follows:

$$\begin{array}{ll} F \leq_* G & := F \to G \\ F \leq_{\kappa \to \kappa'}^{H, \boldsymbol{H}} G & := \forall X^\kappa \forall Y^\kappa.\, X \leq_\kappa HY \to FX \leq_{\kappa'}^{\boldsymbol{H}} GY \end{array}$$

Note that, in the second line, H has kind $\kappa \to \kappa$. Similarly, we define another constructor ${}^{(-)}\leq_\kappa \colon (\boldsymbol{\kappa} \to \boldsymbol{\kappa}) \to \kappa \to \kappa \to *$, where the base case is the same as before, hence no ambiguity with the notation arises.

$$\begin{array}{ll} F \leq_* G & := F \to G \\ F\ {}^{H, \boldsymbol{H}}\!\leq_{\kappa \to \kappa'} G & := \forall X^\kappa \forall Y^\kappa.\, HX \leq_\kappa Y \to FX\ {}^{\boldsymbol{H}}\!\leq_{\kappa'} GY \end{array}$$

As an example, for $F, G, H : \kappa 1$, one has

$$\begin{array}{ll} F \leq_{\kappa 1}^{H} G = \forall A \forall B.\, (A \to HB) \to FA \to GB, \\ F\ {}^{H}\!\leq_{\kappa 1} G = \forall A \forall B.\, (HA \to B) \to FA \to GB. \end{array}$$

If $\boldsymbol{H}$ is just a vector of identity constructors Id, we omit the superscript and simply write $F \leq_\kappa G$. In section 5.1, these notions will be expressed in terms of Kan extensions.

Monotonicity. We can define monotonicity $\mathsf{mon}_\kappa : \kappa \to *$ for kind κ by

$$\mathsf{mon}_\kappa\, F := F \leq_\kappa F,$$

hence $\mathsf{mon}_\kappa\, F$ is a type, seen as the proposition asserting that F is monotone. This notion does not enter the formulation of system MIt^ω, but many applications. We omit the subscripted kind κ when clear from the context, as in the definition of the following basic *monotonicity witnesses*, i. e., closed terms whose type is some $\mathsf{mon}\, F$, which we require for some examples.

$$\begin{array}{lll}
\mathsf{pair} & : \mathsf{mon}(\lambda A \lambda B.A \times B) & := \lambda f \lambda g \lambda p.\, \langle f\,(p.0),\, g\,(p.1)\rangle \\
\mathsf{fork} & : \mathsf{mon}(\lambda A.A \times A) & := \lambda f.\, \mathsf{pair}\, f f \\
\mathsf{either} & : \mathsf{mon}(\lambda A \lambda B.A + B) & := \lambda f \lambda g \lambda x.\, \mathsf{case}\,(x,\, a.\, \mathsf{inl}\,(f\, a),\, b.\, \mathsf{inr}\,(g\, b)) \\
\mathsf{maybe} & : \mathsf{mon}(\lambda A.1 + A) & := \mathsf{either}\ \mathsf{id}
\end{array}$$

3.2 System MIt$^\omega$

Now we are ready to define generalized Mendler-style iteration and coiteration, which specialize to ordinary Mendler-style iteration and coiteration in the case of (co)inductive types, and to a scheme encompassing "generalized folds" [7] and a dual scheme for coinductive constructors of rank 2. This gives an extension of Mendler's system [15] to finite kinds. The generalized scheme for coinductive constructors is a new principle of programming with non-wellfounded datatypes.

The system MIt^ω is given as an extension of F^ω by wellkinded constructor constants μ_κ and ν_κ, welltyped term constants $\mathsf{in}_\kappa, \mathsf{GIt}_\kappa, \mathsf{out}_\kappa$ and GCoit_κ for every kind κ, and new term reduction rules.

Inductive constructors. Let $\kappa = \boldsymbol{\kappa} \to *$ and $\boldsymbol{\kappa}' = \boldsymbol{\kappa} \to \boldsymbol{\kappa}$.

$$\begin{array}{lll}
\text{Formation.} & \mu_\kappa & : (\kappa \to \kappa) \to \kappa \\
\text{Introduction.} & \mathsf{in}_\kappa & : \forall F^{\kappa\to\kappa}.\, F\,(\mu_\kappa F) \subseteq_\kappa \mu_\kappa F \\
\text{Elimination.} & \mathsf{GIt}_\kappa & : \forall F^{\kappa\to\kappa} \forall \boldsymbol{H}^{\boldsymbol{\kappa}'} \forall G^\kappa.\, (\forall X^\kappa.\, X \leq^{\boldsymbol{H}} G \to F\, X \leq^{\boldsymbol{H}} G) \to \mu_\kappa F \leq^{\boldsymbol{H}} G \\
\text{Reduction.} & \mathsf{GIt}_\kappa\, s\, \boldsymbol{f}\, (\mathsf{in}_\kappa\, t) & \longrightarrow_\beta s\, (\mathsf{GIt}_\kappa\, s)\, \boldsymbol{f}\, t
\end{array}$$

with $|\boldsymbol{f}| = |\boldsymbol{\kappa}|$.

Coinductive constructors. Let $\kappa = \boldsymbol{\kappa} \to *$ and $\boldsymbol{\kappa}' = \boldsymbol{\kappa} \to \boldsymbol{\kappa}$.

$$\begin{array}{lll}
\text{Formation.} & \nu_\kappa & : (\kappa \to \kappa) \to \kappa \\
\text{Elimination.} & \mathsf{out}_\kappa & : \forall F^{\kappa\to\kappa}.\, \nu_\kappa F \subseteq_\kappa F\,(\nu_\kappa F) \\
\text{Introduction.} & \mathsf{GCoit}_\kappa & : \forall F^{\kappa\to\kappa} \forall \boldsymbol{H}^{\boldsymbol{\kappa}'} \forall G^\kappa.\, (\forall X^\kappa.\, G\ {}^{\boldsymbol{H}}\!\leq X \to G\ {}^{\boldsymbol{H}}\!\leq F\, X) \to G\ {}^{\boldsymbol{H}}\!\leq \nu_\kappa F \\
\text{Reduction.} & \mathsf{out}_\kappa\, (\mathsf{GCoit}_\kappa\, s\, \boldsymbol{f}\, t) & \longrightarrow_\beta s\, (\mathsf{GCoit}_\kappa\, s)\, \boldsymbol{f}\, t
\end{array}$$

with $|\boldsymbol{f}| = |\boldsymbol{\kappa}|$.

Notice that for *every* constructor F of kind $\kappa \to \kappa$, $\mu_\kappa F$ is a constructor of kind κ. In Mendler's original system [15] as well as its variant for the treatment

of primitive (co-)recursion [14], always positivity of F is required which is a very natural concept in the case $\kappa = *$. However, for higher kinds, there does not exist such a canonical syntactic restriction. Anyway, in [17] it has been observed that, in order to prove strong normalization, *there is no need for the restriction to positive inductive types*—an observation which has been the cornerstone for the treatment of monotone inductive types in [12] and becomes even more useful for our higher-order nested datatypes.

As for F^ω, denote the term closure of the reduction rules by $\longrightarrow$ and its transitive closure by $\longrightarrow^+$.

3.3 Mendler-style (Co)Iteration for (Co)Inductive Types

In the case $\kappa = *$, the rules for μ_κ and ν_κ match with Mendler's [15], except for our removal of the positivity condition and our choice of Curry-style typing:

Inductive types.

Formation.	μ_*	$: (* \to *) \to *$
Introduction.	in_*	$: \forall F^{*\to *}.\, F\,(\mu_* F) \to \mu_* F$
Elimination.	GIt_*	$: \forall F^{*\to *} \forall Y^*.\, (\forall X^*.\, (X \to Y) \to F\,X \to Y) \to \mu_* F \to Y$
Reduction.	$\mathsf{GIt}_*\, s\, (\mathsf{in}_*\, t)$	$\longrightarrow_\beta s\, (\mathsf{GIt}_*\, s)\, t$

Coinductive types.

Formation.	ν_*	$: (* \to *) \to *$
Elimination.	out_*	$: \forall F^{*\to *}.\, \nu_* F \to F\,(\nu_* F)$
Introduction.	GCoit_*	$: \forall F^{*\to *} \forall Y^*.\, (\forall X^*.\, (Y \to X) \to Y \to F\,X) \to Y \to \nu_* F$
Reduction.	$\mathsf{out}_*\, (\mathsf{GCoit}_*\, s\, t)$	$\longrightarrow_\beta s\, (\mathsf{GCoit}_*\, s)\, t$

Relation to general recursion. Typed functional programming languages like ML and Haskell use recursive types instead of inductive and coinductive types and general recursion instead of strongly normalizing restrictions such as Mendler (co)iteration. General recursion can be introduced via a fixed-point combinator

$$\mathsf{fix} : \forall A.(A \to A) \to A$$
$$\mathsf{fix}\ s \longrightarrow s\, (\mathsf{fix}\ s),$$

from which the more common $\texttt{let rec}\ f = r\ \texttt{in}\ t$ can be defined as $\texttt{let}\ f = \mathsf{fix}\ (\lambda f.\, r)\ \texttt{in}\ t$. A nice aspect of Mendler (co)iteration is that the reduction behaviour GIt_* and GCoit_* is almost identical to the one of fix. The only difference is that unfolding of GIt resp. GCoit is controlled by a guard "in" resp. "out", which gets removed in the reduction step. Guarded unfolding of recursion is essential to strong normalization; similar setups can be found in other systems which facilitate *type-based termination*, e.g. [9, 1, 4].

In some sense GIt and GCoit are just restricted versions of fix, i. e., each rank-1 MIt^ω program translates (requiring minimal changes) into a Haskell program

with the same meaning. For higher kinds κ, GIt_κ and GCoit_κ are not typable in the Hindley-Milner type systems of Haskell 98 and ML, but their reduction behaviour is still included in the one of fix. This suggests that one can code most naturally with GIt and GCoit, which we will demonstrate in the next subsection by some examples involving so-called *nested* or *heterogeneous datatypes*.

3.4 Programming with (Co)Inductive Constructors of Rank 2

Nested or heterogenous datatypes, that is inductive and coinductive constructors of rank 2, arise in our system as applications of $\mu_{\kappa 1}$ and $\nu_{\kappa 1}$ (recall that $\kappa 1 = * \to *$ and $\kappa 2 = \kappa 1 \to \kappa 1$). We obtain the following instances from the general definitions.

Inductive constructors of rank 2.

$$\begin{array}{ll}
\text{Formation.} & \mu_{\kappa 1} \quad : \kappa 2 \to \kappa 1 \\
\text{Introduction.} & \mathsf{in}_{\kappa 1} \quad : \forall F^{\kappa 2} \forall A.\, F\,(\mu_{\kappa 1} F)\,A \to \mu_{\kappa 1} F A \\
\text{Elimination.} & \mathsf{GIt}_{\kappa 1} : \forall F^{\kappa 2} \forall H^{\kappa 1} \forall G^{\kappa 1}.\, (\forall X^{\kappa 1}.\, X \leq^H G \to F\,X \leq^H G) \to \mu_{\kappa 1} F \leq^H G \\
\text{Reduction.} & \mathsf{GIt}_{\kappa 1}\, s\, f\, (\mathsf{in}_{\kappa 1}\, t) \longrightarrow_\beta s\, (\mathsf{GIt}_{\kappa 1}\, s)\, f\, t
\end{array}$$

Coinductive constructors of rank 2.

$$\begin{array}{ll}
\text{Formation.} & \nu_{\kappa 1} \quad : \kappa 2 \to \kappa 1 \\
\text{Elimination.} & \mathsf{out}_{\kappa 1} \quad : \forall F^{\kappa 2} \forall A.\, \nu_{\kappa 1} F A \to F\,(\nu_{\kappa 1} F) A \\
\text{Introduction.} & \mathsf{GCoit}_\kappa : \forall F^{\kappa 2} \forall H^{\kappa 1} \forall G^{\kappa 1}.\, (\forall X^{\kappa 1}.\, G\ {}^H\!\leq X \to G\ {}^H\!\leq F\,X) \to G\ {}^H\!\leq \nu_{\kappa 1} F \\
\text{Reduction.} & \mathsf{out}_{\kappa 1}\, (\mathsf{GCoit}_{\kappa 1}\, s\, f\, t) \longrightarrow_\beta s\, (\mathsf{GCoit}_{\kappa 1}\, s)\, f\, t
\end{array}$$

An example of a structure which can be modeled by a nested datatype is lists of length 2^n, which are called *powerlists* [5] or *perfectly balanced, binary leaf trees* [10]. In our system, they are represented by the type transformer $\mathsf{PList} := \mu_{\kappa 1}\, \mathsf{PListF}$ where $\mathsf{PListF} : \kappa 2 := \lambda F \lambda A.\, A + F(A \times A)$. The data constructors are given by

$$\begin{array}{ll}
\mathsf{zero} : \forall A.\, A \to \mathsf{PList}\, A & := \lambda a.\, \mathsf{in}_{\kappa 1}(\mathsf{inl}\, a) \\
\mathsf{succ} : \forall A.\, \mathsf{PList}(A \times A) \to \mathsf{PList}\, A & := \lambda l.\, \mathsf{in}_{\kappa 1}(\mathsf{inr}\, l)
\end{array}$$

Assume a type Nat of natural numbers with addition "+" and multiplication "×", both written infix. Suppose we want to define a function $\mathsf{sum} : \mathsf{PList}\,\mathsf{Nat} \to \mathsf{Nat}$ which sums up all elements of a powerlist by iteration over its structure. The case $\mathsf{sum}\,(\mathsf{succ}\ t)$ imposes some challenge, since sum cannot be directly applied to $t : \mathsf{PList}(\mathsf{Nat} \times \mathsf{Nat})$. The solution is to define a more general function sum' by polymorphic recursion, which has the following behaviour.

$$\begin{array}{l}
\mathsf{sum}' : \forall A.\, (A \to \mathsf{Nat}) \to \mathsf{PList}\, A \to \mathsf{Nat} \\
\mathsf{sum}'\ f\, (\mathsf{zero}\ a) \longrightarrow^+ f\, a \\
\mathsf{sum}'\ f\, (\mathsf{succ}\ l) \longrightarrow^+ \mathsf{sum}'\, (\lambda p.\, f\,(p.0) + f\,(p.1))\, l
\end{array}$$

Here, the iteration process builds up a continuation f which in the end sums up the contents packed into a. From sum', the summation function is obtained by $\mathsf{sum} := \mathsf{sum}'\ \mathsf{id}$.

The system MIt^ω has been designed so that functions like sum' can be defined directly via generalized iteration. In our case, use the instantiations $F := \mathsf{PListF}$ and $G := H := \lambda\_.\,\mathsf{Nat}$ and define:

$$\begin{array}{lcl} \mathsf{sum}' & : & \mu_{\kappa 1} F \leq^H G \\ \mathsf{sum}' & := & \mathsf{GIt}_{\kappa 1}\ \lambda sum'\lambda f\lambda x.\,\mathsf{case}\,(x,\ a.\,f\,a, \\ & & \qquad\qquad\qquad\qquad l.\,sum'\,(\lambda p.\,f\,(p.0) + f\,(p.1))\,l) \end{array}$$

The postulated reduction behaviour is verified by a simple calculation.

For another example consider the non-wellfounded version of perfectly balanced, binary (node-labelled) trees. They are represented by the type transformer $\mathsf{BTree} := \nu_{\kappa 1}\,\mathsf{BTreeF}$ where $\mathsf{BTreeF} : \kappa 2 := \lambda F\lambda A.\,A \times F(A \times A)$. The data destructors are

$$\begin{array}{lll} \mathsf{root} : \forall A.\ \mathsf{BTree}\,A \to A & := \lambda t.\,(\mathsf{out}_{\kappa 1} t).0 \\ \mathsf{subs} : \forall A.\ \mathsf{BTree}\,A \to \mathsf{BTree}(A \times A) & := \lambda t.\,(\mathsf{out}_{\kappa 1} t).1 \end{array}$$

We want to define the tree $\mathsf{nats} : \mathsf{BTree}\,\mathsf{Nat}$ filled with natural numbers starting with 1 breadth-first left-first. A more general function $\mathsf{nats}' : \forall A.\ (\mathsf{Nat} \to A) \to (\mathsf{Nat} \to \mathsf{BTree}\,A)$ with the reduction behaviour

$$\begin{array}{l} \mathsf{root}(\mathsf{nats}'\ f\,n) \longrightarrow^+ f\,n \\ \mathsf{subs}(\mathsf{nats}'\ f\,n) \longrightarrow^+ \mathsf{nats}'\,(\lambda m.\,\langle f\,(2 \times m), f\,(2 \times m + 1)\rangle)\,n \end{array}$$

is definable as a generalized coiteration by

$$\mathsf{nats}' := \mathsf{GCoit}_{\kappa 1}\ \lambda nats'\lambda f\lambda n.\ \langle f\,n,\ nats'\,(\lambda m.\,\langle f\,(2 \times m), f\,(2 \times m + 1)\rangle)\,n\rangle$$

choosing $F := \mathsf{BTreeF}$, $G := H := \lambda\_.\,\mathsf{Nat}$. To obtain nats, one sets $\mathsf{nats} := \mathsf{nats}'\ \mathsf{id}\ 1$.

Triangles. The dual of substitution for variables in a term or non-wellfounded term is redecoration of a non-wellfounded or wellfounded decorated tree, cf. [19]. An interesting and intuitive example of decorated tree types arising from a rank-2 coinductive constructor are triangles. Define

$$\begin{array}{lll} \mathsf{TriF} := \lambda E\lambda F^{\kappa 1}\lambda A.\ A \times F(E \times A) & : \star \to \kappa 2 \\ \mathsf{Tri} \ := \lambda E.\ \nu_{\kappa 1}(\mathsf{TriF}\,E) & : \star \to \kappa 1 \end{array}$$

Then $\mathsf{Tri}\,EA$ is the type of triangular tables of the sort

$$\begin{array}{r} A\ E\ E\ E\ E\ \ldots \\ A\ E\ E\ E\ \ldots \\ \hline \underline{A}\ E\ E\ \ldots \\ A\ E\ \ldots \\ A\ \ldots \end{array}$$

Define also destructors and a monotonicity witness:

$$\begin{array}{lcl}\mathsf{top} := \lambda t.\ (\mathsf{out}_{\kappa 1}\, t).0 & : & \forall E \forall A.\, \mathsf{Tri}\, E A \to A \\ \mathsf{rest} := \lambda t.\ (\mathsf{out}_{\kappa 1}\, t).1 & : & \forall E \forall A.\, \mathsf{Tri}\, E A \to \mathsf{Tri}\, E (E \times A)\end{array}$$

$$\mathsf{tri} := \mathsf{GCoit}_{\kappa 1}\ \lambda map \lambda f \lambda x.\, \langle f\, (\mathsf{top}\ x),\ map\, (\mathsf{pair}\ \mathsf{id}\ f)\, (\mathsf{rest}\ x)\rangle : \forall E.\, \mathsf{mon}_{\kappa 1}(\mathsf{Tri}\, E)$$

Redecoration is an operation dual to substitution that takes a redecoration rule f (an assignment of B-decorations to A-decorated trees) and an A-decorated tree t, and returns a B-decorated tree t'. The return tree t' is obtained from t by B-redecorating every node based on the A-decorated subtree it roots as instructed by the redecoration rule. For streams, for instance $\mathsf{redec} : \forall A \forall B.\, (\mathsf{Str}\, A \to B) \to \mathsf{Str}\, A \to \mathsf{Str}\, B)$ takes $f : \mathsf{Str}\, A \to B$ and $t : \mathsf{Str}\, A$ and returns $\mathsf{redec}\ \ f\ t$, which is a B-stream obtained from t by replacing each of its elements by what f assigns to the substream this element heads. Triangles are a generalization of streams much in the same way as lambda-terms differ from terms in the universal algebra style signature with one binary and one unary operator. For triangles, redecoration works as follows: In the triangle pictured above, the underlined A gets replaced by the B assigned by the redecoration rule to the subtriangle cut out by the horizontal line. This is straightforward to define using GCoit:

$$\begin{array}{lcl}\mathsf{lift} & : & \forall E \forall A \forall B.\, (\mathsf{Tri}\, E A \to B) \to \mathsf{Tri}\, E (E \times A) \to E \times B \\ \mathsf{lift} & := & \lambda f \lambda y.\, \langle (\mathsf{top}\ y).0,\ f\, (\mathsf{tri}\, (\lambda p.\ p.1)\, y)\rangle \\ \mathsf{redec} : & & \forall E \forall A \forall B.\, (\mathsf{Tri}\, E A \to B) \to \mathsf{Tri}\, E A \to \mathsf{Tri}\, E B \\ \mathsf{redec} & := & \mathsf{GCoit}_{\kappa 1}\, \lambda redec \lambda f \lambda x.\, \langle f\, x, redec\, (\mathsf{lift}\, f)\, (\mathsf{rest}\, x)\rangle\end{array}$$

4 A Subsystem of Generalized Folds

System MIt^{ω} facilitates generalized iteration and coiteration in the style of general recursion. In this section we investigate a subsystem of MIt^{ω} which has the flavour of *generalized folds* à la Bird and Paterson [7].

Given $\kappa = \boldsymbol{\kappa} \to *$, $\boldsymbol{\kappa'} = \boldsymbol{\kappa} \to \boldsymbol{\kappa}$, $F : \kappa \to \kappa$, $\boldsymbol{H} : \boldsymbol{\kappa'}$ and $G : \kappa$, we want to find a term r of type $\mu_\kappa F \leq^{\boldsymbol{H}} G$. Of course, this is done by finding a "step term"

$$\hat{s} : \forall X^{\kappa}.\, X \leq^{\boldsymbol{H}} G \to F\, X \leq^{\boldsymbol{H}} G$$

and setting $r := \mathsf{GIt}_{\kappa} \hat{s}$. The typical way to construct $\hat{s}$ would be a λ-abstraction of an argument of type $X \leq^{\boldsymbol{H}} G$. From this, one would construct a term of type $F\, X \leq^{\boldsymbol{H}} G$. The idea now is to choose some appropriate constructor F' of kind $\kappa \to \kappa$ and to construct terms of types $F\, X \leq^{\boldsymbol{H}} F'\, G$ and $F'\, G \subseteq_{\kappa} G$. We must not forget that this has to be done for arbitrary $X : \kappa$. Certainly, we cannot expect a term of type $F'\, G \subseteq_{\kappa} G$ for arbitrary G instead of just the given one. But, as the examples will show, we can safely require—instead of the other term of type $F\, X \leq^{\boldsymbol{H}} F'\, G$ for all $X : \kappa$ (that may use the argument of type $X \leq^{\boldsymbol{H}} G$)—to have a term of type $\forall X^{\kappa} \forall Y^{\kappa}. X \leq^{\boldsymbol{H}} Y \to F\, X \leq^{\boldsymbol{H}} F'\, Y$.

To sum up, we have the following derived rule for $F, F' : \kappa \to \kappa$, $G : \kappa$ and $\boldsymbol{H} : \boldsymbol{\kappa}'$:

$$\frac{\ell : \forall X^\kappa \forall Y^\kappa.\ X \leq^{\boldsymbol{H}} Y \to F\,X \leq^{\boldsymbol{H}} F'\,Y \qquad s : F'\,G \subseteq_\kappa G}{\mathsf{gfold}^+_\kappa(\ell, s) : \mu_\kappa F \leq^{\boldsymbol{H}} G},$$

where we set $\mathsf{gfold}^+_\kappa(\ell, s) := \mathsf{GIt}_\kappa\Big(\lambda i \lambda \boldsymbol{f} \lambda t.\ s\,(\ell\, i\, \boldsymbol{f} t)\Big)$ with $|\boldsymbol{f}| = |\boldsymbol{\kappa}|$ (which we will assume throughout).

We get the following reduction behaviour:

$$\mathsf{gfold}^+_\kappa(\ell, s)\boldsymbol{f}(\mathsf{in}_\kappa t) \longrightarrow^+ s(\ell\, \mathsf{gfold}^+_\kappa(\ell, s)\, \boldsymbol{f} t).$$

The present development can easily be dualized and yields, for $F, F' : \kappa \to \kappa$, $G : \kappa$ and $\boldsymbol{H} : \boldsymbol{\kappa}'$, setting $\mathsf{gunfold}^+_\kappa(\ell, s) := \mathsf{GCoit}_\kappa\Big(\lambda i \lambda \boldsymbol{f} \lambda t.\ell\, i \boldsymbol{f}(s\, t)\Big)$ the derived typing rule

$$\frac{\ell : \forall X^\kappa \forall Y^\kappa . X \overset{\boldsymbol{H}}{\leq} Y \to F'\,X \overset{\boldsymbol{H}}{\leq} F\,Y \qquad s : G \subseteq_\kappa F'\,G}{\mathsf{gunfold}^+_\kappa(\ell, s) : G \overset{\boldsymbol{H}}{\leq} \nu_\kappa F}$$

and the reduction behaviour

$$\mathsf{out}_\kappa(\mathsf{gunfold}^+_\kappa(\ell, s)\boldsymbol{f} t) \longrightarrow^+ \ell\, \mathsf{gunfold}^+_\kappa(\ell, s)\, \boldsymbol{f}(s\, t).$$

The symmetry of GIt_κ and GCoit_κ in the sense of same right-hand sides of their associated reduction rules (except the name of the (co)iterator) is now broken.

In the following sections we study some applications of gfold^+ and $\mathsf{gunfold}^+$.

4.1 (Co)Iteration on Monotone (Co)Inductive Constructors

First consider gfold^+_κ and $\mathsf{gunfold}^+_\kappa$ for $\boldsymbol{H}$ a vector of identity constructors and $F' = F$. Note that in this case, the first argument ℓ receives, for μ_κ as well as for ν_κ, the type

$$\forall X^\kappa \forall Y^\kappa . X \leq_\kappa Y \to F\,X \leq_\kappa F\,Y \ \equiv\ \mathsf{mon}_{\kappa\to\kappa} F.$$

By replacing ℓ by m, and using new names fold_κ and unfold_κ instead of gfold^+_κ and $\mathsf{gunfold}^+_\kappa$, respectively, we arrive at

$$\frac{m : \mathsf{mon}_{\kappa\to\kappa} F \qquad s : F\,G \subseteq_\kappa G}{\mathsf{fold}_\kappa(m, s) : \mu_\kappa F \leq_\kappa G} \qquad \frac{m : \mathsf{mon}_{\kappa\to\kappa} F \qquad s : G \subseteq_\kappa F\,G}{\mathsf{unfold}_\kappa(m, s) : G \leq_\kappa \nu_\kappa F}$$

with reduction behaviour

$$\begin{array}{ll} \mathsf{fold}_\kappa(m, s)\ \boldsymbol{f}\ (\mathsf{in}_\kappa\, t) & \longrightarrow^+ s\,(m\, \mathsf{fold}_\kappa(m, s)\, \boldsymbol{f}\, t) \\ \mathsf{out}_\kappa\ (\mathsf{unfold}_\kappa(m, s)\, \boldsymbol{f}\, t) & \longrightarrow^+ m\, \mathsf{unfold}_\kappa(m, s)\, \boldsymbol{f}\ (s\, t) \end{array}$$

The first interesting example for the use of fold_κ and unfold_κ is the preservation of monotonicity under the formation of least and greatest fixed-points: Define $\mathsf{M}^\mu_\kappa(m) := \mathsf{fold}_\kappa(m, \mathsf{in}_\kappa)$ and $\mathsf{M}^\nu_\kappa(m) := \mathsf{unfold}_\kappa(m, \mathsf{out}_\kappa)$. Then we have

$$\frac{m : \mathsf{mon}_{\kappa\to\kappa} F}{\mathsf{M}^\mu_\kappa(m) : \mathsf{mon}_\kappa(\mu_\kappa F)} \qquad \frac{m : \mathsf{mon}_{\kappa\to\kappa} F}{\mathsf{M}^\nu_\kappa(m) : \mathsf{mon}_\kappa(\nu_\kappa F)}$$

with reduction behaviour

$$\begin{array}{l}\mathsf{M}^{\mu}_{\kappa}(m)\,\boldsymbol{f}\,(\mathsf{in}_{\kappa}\,t) \;\longrightarrow^{+}\; \mathsf{in}_{\kappa}\,(m\;\mathsf{M}^{\mu}_{\kappa}(m)\,\boldsymbol{f}\,t)\\ \mathsf{out}_{\kappa}\,(\mathsf{M}^{\nu}_{\kappa}(m)\,\boldsymbol{f}\,t) \longrightarrow^{+}\; m\;\mathsf{M}^{\nu}_{\kappa}(m)\,\boldsymbol{f}\,(\mathsf{out}_{\kappa}\,t)\end{array}$$

The first rule now, unlike [2], justifies, modulo the pattern-matching mechanism, the definitions of $nest : \mathsf{mon}(\mu_{\kappa 1}\lambda F\lambda A.1 + A \times F(A \times A))$ and $host : \mathsf{mon}(\mu_{\kappa 1}\lambda F\lambda A.1 + A \times F(A \times FA))$ in [7, Examples 3.2,3.3], and, more generally, the construction of map terms described in [10]. We spell this out for the case of powerlists. Monotonicity of PList is established by

$$\begin{array}{l}\mathsf{plistf} : \mathsf{mon}\,\mathsf{PListF} := \lambda s\lambda f.\,\mathsf{either}\,f\,(s\,(\mathsf{fork}\,f))\\ \mathsf{plist}\;\; : \mathsf{mon}\,\mathsf{PList}\;\; := \mathsf{M}^{\mu}_{\kappa 1}(\mathsf{plistf})\end{array}$$

It is easy to see that $\mathsf{plist}\,f\,(\mathsf{zero}\,t)$ and $\mathsf{zero}\,(f\,t)$ have a common $\longrightarrow^{+}$-reduct. The same holds for $\mathsf{plist}\,f\,(\mathsf{succ}\,t)$ and $\mathsf{succ}\,(\mathsf{plist}\,(\mathsf{fork}\,f)\,t)$.[1]

For a "coinductive example", note that the definition of the monotonicity witness tri in Section 3.4 has the same normal form as $\mathsf{M}^{\nu}_{\kappa 1}(\mathsf{trif})$ for the canonical term $\mathsf{trif} := \lambda i\lambda f.\,\mathsf{pair}\,f\,(i\,(\mathsf{pair}\,\mathsf{id}\,f)) : \forall E.\,\mathsf{mon}_{\kappa 2}(\mathsf{TriF}\,E)$.

Our present notion of (co)iteration for monotone (co)inductive constructors differs from [2] not in the notion of monotonicity, but in the type of s which is now based on $\subseteq_{\kappa}$ instead of $\leq_{\kappa}$ in order to fit with the same change in the type of in_{κ} and out_{κ}. The crucial point for the preservation of monotonicity is that the result types of fold_{κ} and unfold_{κ} are still formed with the help of $\leq_{\kappa}$.

4.2 Generalized Iteration for Certain Sums

A typical application of $\mathsf{gfold}^{+}_{\kappa}$ where F and F' differ, is as follows: The constructors $F, F' : \kappa \to \kappa$ arise as sums $\lambda X^{\kappa}.J + F_0\,X$, $\lambda X^{\kappa}.G + F_0\,X$ for some constructors $J, G : \kappa$ and $F_0 : \kappa \to \kappa$, none of them depending on X. With $\boldsymbol{H} : \boldsymbol{\kappa'}$, we get the derived typing rule

$$\frac{\ell_0 : J \leq^{\boldsymbol{H}} G \qquad m : \forall X^{\kappa}\forall Y^{\kappa}.X \leq^{\boldsymbol{H}} Y \to F_0\,X \leq^{\boldsymbol{H}} F_0\,Y \qquad s_0 : F_0\,G \subseteq_{\kappa} G}{\mathsf{sgfold}_{\kappa}(\ell_0, m, s) : \mu_{\kappa}F \leq^{\boldsymbol{H}} G}.$$

where

$$\begin{array}{lll}\ell & : & \forall X^{\kappa}\forall Y^{\kappa}.\; X \leq^{\boldsymbol{H}} Y \to (J + F_0X) \leq^{\boldsymbol{H}} (G + F_0Y)\\ \ell & := & \lambda i\lambda\boldsymbol{f}\lambda t.\,\mathsf{case}\,\Big(t,\; x.\;\mathsf{inl}\,(\ell_0\boldsymbol{f}x),\; y.\;\mathsf{inr}\,(m\,i\boldsymbol{f}y)\Big),\\ s & : & (G + F_0\,G) \subseteq G\\ s & := & \lambda t.\,\mathsf{case}\,(t,\; x.\,x,\; y.\,s_0\,y)\\ \mathsf{sgfold}_{\kappa}(\ell_0, m, s_0) & := & \mathsf{gfold}^{+}_{\kappa}(\ell, s)\end{array}$$

[1] In the functional programming language Haskell, these observations would be made into a pattern-based definition of plist.

with $|\boldsymbol{f}| = |\boldsymbol{\kappa}|$. Since no permutative conversions are included in system MIt^ω, the reduction behaviour is not perspicuous enough, whence we prefer to define sgfold_κ directly via GIt_κ by

$$\mathsf{sgfold}_\kappa(\ell_0, m, s_0) := \mathsf{GIt}_\kappa\Big(\lambda i \lambda \boldsymbol{f} \lambda t.\mathsf{case}\,\Big(t,\, x.\,\ell_0 \boldsymbol{f} x,\, y.\, s_0(m\, i\, \boldsymbol{f} y)\Big)\Big)$$

which gives the reduction behaviour

$$\mathsf{sgfold}_\kappa(\ell_0, m, s_0)\boldsymbol{f}(\mathsf{in}_\kappa t) \longrightarrow^+ \mathsf{case}\,\Big(t,\, x.\,\ell_0 \boldsymbol{f} x,\, y.\, s_0\Big(m\,\mathsf{sgfold}_\kappa(\ell_0, m, s_0)\,\boldsymbol{f} y\Big)\Big).$$

Apparently, this expresses that ℓ_0 determines the outcome in the left case, and that in the case to the right, a generalization of iteration on monotone inductive constructors is used since we take into account the $\boldsymbol{H}$.

Higher-order representation of de Bruijn terms. Bird & Paterson [8] and Altenkirch & Reus [3] have shown that nameless untyped λ-terms can be represented by a heterogeneous datatype. As in the system GMIC of [2], this type is obtained in MIt^ω as the least fixed point of the monotone rank-2 constructor LamF.

$$\begin{array}{ll}
\mathsf{LamF} : \kappa 2 & := \lambda F \lambda A.\, A + (FA \times FA + F(1+A)) \\
\mathsf{lamf} : \mathsf{mon}\,\mathsf{LamF} & := \lambda s \lambda f.\ \mathsf{either}\, f\,\Big(\mathsf{either}\,(\mathsf{fork}\,(s\, f))\,(s\,(\mathsf{maybe}\ f))\Big)
\end{array}$$

The type $\mathsf{Lam}\, A$ again represents all de Bruijn terms with free variables in A, the constructors var, app and abs are simplified w. r. t. [2]. Again, we provide an auxiliary function weak which lifts each variable in a term to provide space for a fresh variable.

$$\begin{array}{lll}
\mathsf{Lam} & : \kappa 1 & := \mu_{\kappa 1}\,\mathsf{LamF} \\
\mathsf{lam} & : \mathsf{mon}\,\mathsf{Lam} & := \mathsf{M}^{\mu}_{\kappa 1}(\mathsf{lamf}) \\
\mathsf{var} & : \forall A.\, A \to \mathsf{Lam}\, A & := \lambda a.\, \mathsf{in}_{\kappa 1}\,(\mathsf{inl}\, a) \\
\mathsf{app} & : \forall A.\, \mathsf{Lam}\, A \to \mathsf{Lam}\, A \to \mathsf{Lam}\, A & := \lambda t_1 \lambda t_2.\, \mathsf{in}_{\kappa 1}\,(\mathsf{inr}\,(\mathsf{inl}\,\langle t_1, t_2\rangle)) \\
\mathsf{abs} & : \forall A.\, \mathsf{Lam}(1+A) \to \mathsf{Lam}\, A & := \lambda r.\, \mathsf{in}_{\kappa 1}\,(\mathsf{inr}\,(\mathsf{inr}\, r)) \\
\mathsf{weak} & : \forall A.\, \mathsf{Lam}\, A \to \mathsf{Lam}(1+A) & := \mathsf{lam}\,(\lambda a.\ \mathsf{inr}\, a)
\end{array}$$

The most natural question on this representation of untyped λ-calculus is the representability of substitution. With $\mathsf{sgfold}_{\kappa 1}$, it is possible to give a direct definition of substitution (the bind or extension operation of the lambda terms monad)

$$\mathsf{subst} : \forall A \forall B.\, (A \to \mathsf{Lam}\, B) \to \mathsf{Lam}\, A \to \mathsf{Lam}\, B \;\equiv\; \mathsf{Lam} \leq^{\mathsf{Lam}} \mathsf{Lam}\,.$$

We proceed as follows, with $J := \mathsf{Id}_*$, $\mathsf{LamF}_0 := \lambda F \lambda A. FA \times FA + F(1+A)$, $G, H := \mathsf{Lam}$:

$$\begin{array}{ll}
\ell_0 : & J \leq^H G \equiv \forall A \forall B.(A \to \mathsf{Lam}\, B) \to A \to \mathsf{Lam}\, B \\
\ell_0 := & \mathsf{id}
\end{array}$$

$$\begin{array}{ll}
s_0 : & \mathsf{LamF}_0\, G \subseteq_{\kappa 1} G \equiv \forall A.\, \mathsf{Lam}\, A \times \mathsf{Lam}\, A + \mathsf{Lam}(1+A) \to \mathsf{Lam}\, A \\
s_0 := & \lambda t.\mathsf{case}\,(t,\ p.\ \mathsf{app}(p.0)(p.1),\ l.\ \mathsf{abs}\, l)
\end{array}$$

$$
\begin{array}{lll}
m & : & \forall F \forall F'.F \leq^H F' \to \mathsf{LamF}_0\, F \leq^H \mathsf{LamF}_0\, F' \\
m & := & \lambda i \lambda f \lambda t.\mathsf{case}\left(t,\, p.\, \mathsf{inl}\langle if(p.0), if(p.1)\rangle,\, l.\, \mathsf{inr}(i\, g\, l)\right) \\
 & & \text{with } g := \lambda x.\mathsf{case}\,(x,\, u.\, \mathsf{var}(\mathsf{inl}\, u),\, a.\, \mathsf{weak}(fa))
\end{array}
$$

$$
\mathsf{subst} := \mathsf{sgfold}_{\kappa 1}(\ell_0, m, s_0)
$$

Note that the formulation of generalized folds in [7] would yield the flattening function (the join or multiplication operation of the monad)

$$
\mathsf{flatten} : \forall A.\ \mathsf{Lam}(\mathsf{Lam}\, A) \to \mathsf{Lam}\, A.
$$

We obtain flattening as special case of substitution by $\mathsf{flatten} := \mathsf{subst}\ \mathsf{id}$.

5 Embedding into System F^ω

In this section, we show how to embed MIt^ω into F^ω. The embedding establishes strong normalization for MIt^ω.

5.1 Kan extensions

For the sake of the embedding of MIt^ω into its subsystem F^ω, we use a syntactic version of Kan extensions, see [16, chapter 10]. Compared with [2], Kan extensions "along" are now defined for all kinds, not just for rank 1.

Right Kan extension along $\boldsymbol{H}$. Let $\kappa = \boldsymbol{\kappa} \to *$ and $\boldsymbol{\kappa}' = \boldsymbol{\kappa} \to \boldsymbol{\kappa}$ and define for $G : \kappa$, $\boldsymbol{H} : \boldsymbol{\kappa}'$ and $\boldsymbol{X} : \boldsymbol{\kappa}$ the type $(\mathsf{Ran}_{\boldsymbol{H}}\, G)\boldsymbol{X}$ by iteration on $|\boldsymbol{\kappa}|$:

$$
\begin{array}{rcl}
\mathsf{Ran}\, G & := & G \\
(\mathsf{Ran}_{H,\boldsymbol{H}}\, G)X\boldsymbol{X} & := & \forall Y^{\kappa_1}.\ X \leq HY \to (\mathsf{Ran}_{\boldsymbol{H}}\ GY)\boldsymbol{X}
\end{array}
$$

Left Kan Extension along $\boldsymbol{H}$. Let again $\kappa = \boldsymbol{\kappa} \to *$ and $\boldsymbol{\kappa}' = \boldsymbol{\kappa} \to \boldsymbol{\kappa}$ and define for $F : \kappa$, $\boldsymbol{H} : \boldsymbol{\kappa}'$ and $\boldsymbol{Y} : \boldsymbol{\kappa}$ the type $(\mathsf{Lan}_{\boldsymbol{H}}\, F)\boldsymbol{Y}$ by iteration on $|\boldsymbol{\kappa}|$:

$$
\begin{array}{rcl}
\mathsf{Lan}\, F & := & F \\
(\mathsf{Lan}_{H,\boldsymbol{H}}\, F)Y\boldsymbol{Y} & := & \exists X^{\kappa_1}.\ HX \leq Y \times (\mathsf{Lan}_{\boldsymbol{H}}\ FX)\boldsymbol{Y}
\end{array}
$$

Proposition 1. *Let $F, G : \boldsymbol{\kappa} \to *$ and $\boldsymbol{H} : \boldsymbol{\kappa} \to \boldsymbol{\kappa}$. The following pairs of types are logically equivalent:*

1. $F \leq^{\boldsymbol{H}} G$ *and* $\forall \boldsymbol{X}^{\boldsymbol{\kappa}}.\ F\boldsymbol{X} \to (\mathsf{Ran}_{\boldsymbol{H}}\, G)\boldsymbol{X}$.
2. $F \overset{\boldsymbol{H}}{\leq} G$ *and* $\forall \boldsymbol{Y}^{\boldsymbol{\kappa}}.\ (\mathsf{Lan}_{\boldsymbol{H}}\, F)\boldsymbol{Y} \to G\,\boldsymbol{Y}$.

Proof. Part 1 requires just a close look at the definition of $\leq^{\boldsymbol{H}}$. Part 2 is only slightly more complicated. □

5.2 Embedding

We can simply define the new constants of MIt^ω in F^ω. Let $\kappa = \boldsymbol{\kappa} \to *$ and $n := |\boldsymbol{\kappa}|$.

$$
\begin{array}{lll}
\mu_\kappa & : & (\kappa \to \kappa) \to \boldsymbol{\kappa} \to * \\
\mu_\kappa & := & \lambda F^{\kappa\to\kappa} \lambda \boldsymbol{X}^{\boldsymbol{\kappa}} \forall \boldsymbol{H}^{\boldsymbol{\kappa}\to\boldsymbol{\kappa}} \forall G^{\kappa}.\ (\forall X^{\kappa}.\, X \leq^{\boldsymbol{H}} G \to FX \leq^{\boldsymbol{H}} G) \to (\mathsf{Ran}_{\boldsymbol{H}}\, G)\boldsymbol{X} \\
\\
\mathsf{GIt}_\kappa & : & \forall F^{\kappa\to\kappa} \forall \boldsymbol{H}^{\boldsymbol{\kappa}\to\boldsymbol{\kappa}} \forall G^{\kappa}.\ (\forall X^{\kappa}.\, X \leq^{\boldsymbol{H}} G \to F\, X \leq^{\boldsymbol{H}} G) \to \mu_\kappa F \leq^{\boldsymbol{H}} G \\
\mathsf{GIt}_\kappa & := & \lambda s \lambda \boldsymbol{f} \lambda r.\, r\, s\, \boldsymbol{f} \\
\\
\mathsf{in}_\kappa & : & \forall F^{\kappa\to\kappa} \forall \boldsymbol{X}^{\boldsymbol{\kappa}}.\ F\,(\mu_\kappa F)\,\boldsymbol{X} \to \mu_\kappa F\, \boldsymbol{X} \\
\mathsf{in}_\kappa & := & \lambda t \lambda s \lambda \boldsymbol{f}.\, s\,(\mathsf{GIt}_\kappa\, s)\, \boldsymbol{f}\, t \\
\\
\nu_\kappa & : & (\kappa \to \kappa) \to \boldsymbol{\kappa} \to * \\
\nu_\kappa & := & \lambda F^{\kappa\to\kappa} \lambda \boldsymbol{Y}^{\boldsymbol{\kappa}} \exists \boldsymbol{H}^{\boldsymbol{\kappa}\to\boldsymbol{\kappa}} \exists G^{\kappa}.\ (\forall X^{\kappa}.\, G\, {}^{\boldsymbol{H}}\!\leq X \to G\, {}^{\boldsymbol{H}}\!\leq FX) \times (\mathsf{Lan}_{\boldsymbol{H}}\, G)\boldsymbol{Y} \\
\\
\mathsf{GCoit}_\kappa & : & \forall F^{\kappa\to\kappa} \forall \boldsymbol{H}^{\boldsymbol{\kappa}\to\boldsymbol{\kappa}} \forall G^{\kappa}.\ (\forall X^{\kappa}.\, G\, {}^{\boldsymbol{H}}\!\leq X \to G\, {}^{\boldsymbol{H}}\!\leq F\, X) \to G\, {}^{\boldsymbol{H}}\!\leq \nu_\kappa F \\
\mathsf{GCoit}_\kappa & := & \lambda s \lambda \boldsymbol{f} \lambda t.\ \ \mathsf{pack}^{n+1}\, \langle s,\, \mathsf{pack}\langle f_1,\, \ldots,\, \mathsf{pack}\langle f_n,\, t\rangle \ldots\rangle\rangle \\
\\
\mathsf{out}_\kappa & : & \forall F^{\kappa\to\kappa} \forall \boldsymbol{Y}^{\boldsymbol{\kappa}}.\ \nu_\kappa F\, \boldsymbol{Y} \to F\,(\nu_\kappa F)\, \boldsymbol{Y} \\
\mathsf{out}_\kappa & := & \lambda r.\ \mathsf{open}\,(r,\, r_1.\, \mathsf{open}\,(r_1,\, r_2.\, \ldots \mathsf{open}\,(r_{n-1},\, r_n.\, \mathsf{open}\,(r_n,\, ft_0. \\
 & & \quad \mathsf{open}\,(ft_0.1,\, ft_1.\, \mathsf{open}\,(ft_1.1,\, ft_2.\, \ldots \mathsf{open}\,(ft_{n-1}.1,\, ft_n. \\
 & & \quad ft_0.0\ (\mathsf{GCoit}_\kappa\ ft_0.0)\ ft_1.0\ \ \ldots ft_n.0\ ft_n.1) \ldots)))) \ldots))
\end{array}
$$

Theorem 1 (Simulation). *With the definitions above, the following reductions take place in* F^ω*:*

$$
\begin{array}{ll}
\mathsf{GIt}_\kappa\, s\, \boldsymbol{f}\, (\mathsf{in}_\kappa\, t) & \longrightarrow^+ s\,(\mathsf{GIt}_\kappa\, s)\, \boldsymbol{f}\, t \\
\mathsf{out}_\kappa\, (\mathsf{GCoit}_\kappa\, s\, \boldsymbol{f}\, t) & \longrightarrow^+ s\,(\mathsf{GCoit}_\kappa\, s)\, \boldsymbol{f}\, t
\end{array}
$$

Proof. By easy computation.

Corollary 1 (Strong Normalization). *System* MIt^ω *is strongly normalizing, i. e., there is no infinite reduction sequence* $r_0 \longrightarrow r_1 \longrightarrow r_2 \longrightarrow \ldots$ *for any typable term* r_0*.*

Proof. Use strong normalization of F^ω and simulation.

Since there are no critical pairs in MIt^ω, reduction is locally confluent; by strong normalization and Newman's Lemma, it is confluent on well-typed terms.

6 Conclusion and Related Work

We have proposed MIt^ω, a system of generalized (co)iteration for arbitrary ranks, which turned out to be a definitional extension of Girard's system F^ω and hence enjoys its good meta-theoretic properties, most notably strong normalization. It combines the ideas of Mendler for (co)inductive types with the notion of generalized folds for inductive functors invented by Bird and Paterson.

MIt^ω has been carefully set up to come with a perspicuous computational behavior, which is very close to general recursion à la `letrec`—a distinctive feature of Mendler-style recursion schemes. For higher ranks, i. e., for the treatment of fixed-points which are themselves type transformers, we described a modified containment relation (via the index $\boldsymbol{H}$) in order to encompass generalized folds, proposed by Bird and Paterson as a means of more elegant definitions of functions operating on nested datatypes.

Therefore, MIt^ω might serve as a basis of a total programming language for nested datatypes. Alternatively, it can be seen as a discipline of programming in existing languages like Haskell which gives termination guarantees for free.

Some related work. The generalized iteration scheme of the present paper is a reformulation of generalized folds in the liberal sense of Sec. 4.1 and 6 of Bird and Paterson [7] that works for all finite kinds (theirs was for rank 2 inductive types). The more specific scheme of our Sect. 4 subsumes generalized folds in the more restricted sense of Sec. 4 of loc.cit. Hinze [10] has investigated *efficient generalized folds*.

Future work. The realm of higher-rank datatypes seems hardly explored. We would greatly appreciate natural examples for rank 3 or higher datatypes on which we could demonstrate the reach of our approach. Further, we seek to extend MIt^ω to cover other recursion schemes on nested datatypes like a form of "generalized primitive recursion". This scheme, and others, would no longer have an operationally faithful embedding into System F^ω.

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The following appendix can be omitted in the final version.

A System F^ω

In the following we present Curry-style system F^ω enriched with binary sums and products, empty and unit type and existential quantification over constructors. Although we choose a human-friendly notation of variables, we actually mean the nameless version à la de Bruijn which identifies α-equivalent terms. (Capture-avoiding) Substitution of an expression e for a variable x in expression f is denoted by $f[x := e]$.

Kinds and rank.

$$\begin{array}{lll} \kappa & ::= & * \mid \kappa \to \kappa' \\ \mathsf{rk}(*) & := & 0 \\ \mathsf{rk}(\kappa \to \kappa') & := & \max(\mathsf{rk}(\kappa) + 1, \mathsf{rk}(\kappa')) \end{array}$$

Constructors. (Denoted by uppercase letters)

$$\begin{array}{c} A, B, F, G ::= X \mid \lambda X^\kappa.F \mid F\,G \mid \forall F^\kappa.\, A \mid \exists F^\kappa.\, A \mid A \to B \\ \mid 0 \mid A + B \mid 1 \mid A \times B \end{array}$$

Equivalence on constructors. Equivalence $F = F'$ for constructors F and F' is given as the compatible closure of the following axiom.

$$(\lambda X.F)\,G =_\beta F[X := G]$$

We identify constructors up to equivalence, which is a decidable relation due to normalization and confluence of simply-typed λ-calculus (where our constructors are the terms and our kinds are the types of that calculus).

Objects (Terms). (Denoted by lowercase letters)

$$\begin{array}{c} r, s, t ::= x \mid \lambda x.t \mid r\,s \mid \mathsf{abort}\,r \mid \mathsf{inl}\,t \mid \mathsf{inr}\,t \mid \mathsf{case}\,(r,\, x.\,s,\, y.\,t) \\ \mid \langle\rangle \mid \langle t_0, t_1\rangle \mid r.0 \mid r.1 \mid \mathsf{pack}\,t \mid \mathsf{open}\,(r,\, x.\,s) \end{array}$$

Contexts. Variables in a context Γ are assumed to be distinct.

$$\Gamma ::= \cdot \mid \Gamma, X^\kappa \mid \Gamma, x : A$$

Judgments. (Simultaneously defined)

$$\begin{array}{ll} \Gamma\ \mathsf{cxt} & \Gamma \text{ is a wellformed context} \\ \Gamma \vdash F : \kappa & F \text{ is a wellformed constructor of kind } \kappa \text{ in context } \Gamma \\ \Gamma \vdash t : A & t \text{ is a wellformed term of type } A \text{ in context } \Gamma \end{array}$$

Wellformed contexts. Γ cxt

$$\frac{}{\cdot \text{ cxt}} \qquad \frac{\Gamma \text{ cxt}}{\Gamma, X^\kappa \text{ cxt}} \qquad \frac{\Gamma \vdash A : *}{\Gamma, x{:}A \text{ cxt}}$$

Wellkinded constructors. $\Gamma \vdash F : \kappa$

$$\frac{X^\kappa \in \Gamma \qquad \Gamma \text{ cxt}}{\Gamma \vdash X : \kappa} \qquad \frac{\Gamma, X^\kappa \vdash F : \kappa'}{\Gamma \vdash \lambda X^\kappa.F : \kappa \to \kappa'} \qquad \frac{\Gamma \vdash F : \kappa \to \kappa' \qquad \Gamma \vdash G : \kappa}{\Gamma \vdash F\,G : \kappa'}$$

$$\frac{\Gamma, X^\kappa \vdash A : *}{\Gamma \vdash \forall X^\kappa.\, A : *} \qquad \frac{\Gamma, X^\kappa \vdash A : *}{\Gamma \vdash \exists X^\kappa.\, A : *} \qquad \frac{\Gamma \vdash A : * \qquad \Gamma \vdash B : *}{\Gamma \vdash A \to B : *}$$

$$\frac{\Gamma \text{ cxt}}{\Gamma \vdash 0 : *} \qquad \frac{\Gamma \vdash A : * \qquad \Gamma \vdash B : *}{\Gamma \vdash A + B : *}$$

$$\frac{\Gamma \text{ cxt}}{\Gamma \vdash 1 : *} \qquad \frac{\Gamma \vdash A : * \qquad \Gamma \vdash B : *}{\Gamma \vdash A \times B : *}$$

Welltyped terms. $\Gamma \vdash t : A$

$$\frac{(x{:}A) \in \Gamma \qquad \Gamma \text{ cxt}}{\Gamma \vdash x : A} \qquad \frac{\Gamma, x{:}A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B} \qquad \frac{\Gamma \vdash r : A \to B \qquad \Gamma \vdash s : A}{\Gamma \vdash r\,s : B}$$

$$\frac{\Gamma, X^\kappa \vdash t : A}{\Gamma \vdash t : \forall X^\kappa.A} \qquad \frac{\Gamma \vdash t : \forall X^\kappa.A \qquad \Gamma \vdash F : \kappa}{\Gamma \vdash t : A[X := F]}$$

$$\frac{\Gamma \vdash t : A[X := F] \qquad \Gamma \vdash F : \kappa}{\Gamma \vdash \mathsf{pack}\, t : \exists X^\kappa.A} \qquad \frac{\Gamma \vdash r : \exists X^\kappa.A \qquad \Gamma, X^\kappa, x{:}A \vdash s : C}{\Gamma \vdash \mathsf{open}\,(r,\, x.\, s) : C}$$

$$\frac{\Gamma \text{ cxt}}{\Gamma \vdash \langle\rangle : 1} \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash B : *}{\Gamma \vdash \mathsf{inl}\, t : A + B} \qquad \frac{\Gamma \vdash t : B \qquad \Gamma \vdash A : *}{\Gamma \vdash \mathsf{inr}\, t : A + B}$$

$$\frac{\Gamma \vdash r : A + B \qquad \Gamma, x{:}A \vdash s : C \qquad \Gamma, y{:}B \vdash t : C}{\Gamma \vdash \mathsf{case}\,(r,\, x.\, s,\, y.\, t) : C} \qquad \frac{\Gamma \vdash r : 0 \qquad \Gamma \vdash C : *}{\Gamma \vdash \mathsf{abort}\, r : C}$$

$$\frac{\Gamma \vdash t_0 : A_0 \qquad \Gamma \vdash t_1 : A_1}{\Gamma \vdash \langle t_0, t_1\rangle : A_0 \times A_1} \qquad \frac{\Gamma \vdash r : A_0 \times A_1 \quad i \in \{0,1\}}{\Gamma \vdash r.i : A_i}$$

Reduction. The one-step reduction relation $t \longrightarrow t'$ between terms t and t' is defined as the closure of the following axioms under all term constructors.

$$\begin{array}{lll}
(\lambda x.t)\, s & \longrightarrow_\beta t[x := s] & \\
\mathsf{case}\,(\mathsf{inl}\, r,\, x.\, s,\, y.\, t) & \longrightarrow_\beta s[x := r] & \\
\mathsf{case}\,(\mathsf{inr}\, r,\, x.\, s,\, y.\, t) & \longrightarrow_\beta t[y := r] & \\
\langle t_0, t_1\rangle.i & \longrightarrow_\beta t_i & \text{if } i \in \{0,1\} \\
\mathsf{open}\,(\mathsf{pack}\, t,\, x.\, s) & \longrightarrow_\beta s[x := t] &
\end{array}$$

We denote the transitive closure of $\longrightarrow$ by $\longrightarrow^+$ and the reflexive-transitive closure by $\longrightarrow^*$.

The defined system is a conservative extension of system F^ω. Reduction is type-preserving, confluent and strongly normalizing.