Slide 1

Termination of Mutually Recursive Functions with Several Arguments by Lexicographic Orderings

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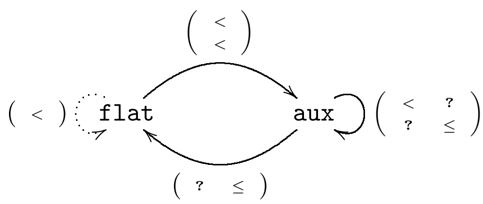
Slide 2

Example

```
fun flat []         = []
  | flat (l::ls)    = aux l ls
and aux  []      ls = flat ls
  | aux  (x::xs) ls = x :: aux xs ls;
```



Slide 3

Size Change Precategory $\mathcal{L}$

- objects: function identifiers $\mathcal{F}$
- morphisms: $\sqcup$-semilattices of size change information

$$\star : \mathcal{L}(g,h) \times \mathcal{L}(f,g) \to \mathcal{L}(f,h)$$

Interpretation $[\![-]\!]$

$$\mathcal{L}(f,g) \ni A \mapsto [\![A]\!] \subseteq \mathcal{D}_g \times \mathcal{D}_f$$

1. $A \sqsubseteq B \Rightarrow [\![A]\!] \subseteq [\![B]\!]$
2. $[\![C \star B]\!] \supseteq [\![C]\!] \circ [\![B]\!]$

Slide 4

Call Graph $\mathcal{C} \subseteq \mathcal{L}$

$$\mathcal{C} : ((f,g) \in \mathcal{F} \times \mathcal{F}) \to \mathcal{P}(\mathcal{L}(f,g))$$

$$f \xrightarrow{A} g \qquad \text{iff} \qquad A \in \mathcal{C}(f,g)$$

Completion $\mathcal{C}^+$

$\mathcal{C}$ complete $:\Longleftrightarrow$ closed under composition

$$\frac{A \in \mathcal{C}(f,g) \qquad B \in \mathcal{C}(g,h)}{B \star A \in \mathcal{C}(f,h)}$$

$\mathcal{C}^+$ $:\Longleftrightarrow$ completion of $\mathcal{C}$

$$f \xrightarrow{A}^{+} g \qquad \text{iff} \qquad A \in \mathcal{C}^+(f,g)$$

Slide 5

Termination ⇓

$$\frac{\forall g,\ f \xrightarrow{A} g,\ v \in \mathcal{D}_g,\ w \in \mathcal{D}_f.\ v\ [\![A]\!]\ w \Rightarrow g@v \Downarrow}{f@w \Downarrow}$$

$$f \Downarrow \;:\Longleftrightarrow \forall v \in \mathcal{D}_f.\ f@v \Downarrow$$

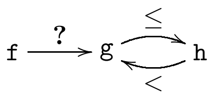
Evaluation Ordering ≪

$$f \xrightarrow{A} g \ \wedge\ v\ [\![A]\!]\ w \ \Rightarrow\ (g, v) \ll (f, w)$$

Prop. If a wellfounded evaluation ordering exists for $\mathcal{C}$, then all functions $f \in \mathcal{F}$ terminate.

Slide 6

Functions with a Single Argument



Slide 7

Single Argument Lattice $\mathbb{L}$

Consists of the three elements $< \sqsubseteq \leq \sqsubseteq \mathbf{?}$.

$\star$	$<$	$\leq$	$\mathbf{?}$
$<$	$<$	$<$	$\mathbf{?}$
$\leq$	$<$	$\leq$	$\mathbf{?}$
$\mathbf{?}$	$\mathbf{?}$	$\mathbf{?}$	$\mathbf{?}$

Interpretation in Wellordered Domain $(D, <)$

$$\begin{aligned} v \llbracket < \rrbracket w \quad &:\Longleftrightarrow \quad v < w \\ v \llbracket \leq \rrbracket w \quad &:\Longleftrightarrow \quad v = w \text{ or } v < w \\ v \llbracket \mathbf{?} \rrbracket w \quad &:\Longleftrightarrow \quad \text{True} \end{aligned}$$

Slide 8

Good Call Graph

$$f \xrightarrow{R}{}^{+} f \quad \text{implies} \quad R = <$$

Lemma.

$$f \xrightarrow{R_0} f_1 \xrightarrow{R_1} \ldots \xrightarrow{R_{n-1}} f_n \xrightarrow{R_n} f \quad \text{implies} \quad R_i \sqsubseteq \leq \text{ for all } i$$

Lemma.

$$f \xrightarrow{R_0} f_1 \xrightarrow{R_1} \ldots \xrightarrow{R_{n-1}} f_n \xrightarrow{R_n} f \quad \text{implies} \quad R_i = < \text{ for some } i$$

Slide 9

Function Orderings

$$\begin{aligned} g \lhd h \quad &:\Longleftrightarrow \quad h \xrightarrow{\leq} g \\ g \preceq h \quad &:\Longleftrightarrow \quad h \longrightarrow g \\ g \approx h \quad &:\Longleftrightarrow \quad g \preceq^+ h \wedge h \preceq^+ g \\ g \prec h \quad &:\Longleftrightarrow \quad g \preceq \setminus \approx h \end{aligned}$$

Theorem. $\lhd^+$ and $\prec^+$ are irreflexive.

Corollary. For a finite $\mathcal{F}$ $\lhd$ and $\prec$ are wellfounded.

Slide 10

Evaluation Ordering $\ll$

$$\begin{aligned} (g, v) \ll (f, w) \;:\Longleftrightarrow \quad & g \prec f \\ \vee \quad & (g \approx f \quad \wedge \quad v < w \\ & \qquad \vee \quad (v = w \quad \wedge g \lhd f)) \end{aligned}$$

Prop. $\ll$ is a wellfounded evaluation ordering.

Slide 11

Functions with Several Arguments

$$\mathsf{ar} : \mathcal{F} \to \mathbb{N}$$

Call Matrices Precategory $\mathcal{L}$

Size change information $(\sigma, a) \in \mathcal{L}(f, g)$

$$\begin{aligned} \sigma &: \mathsf{ar}(g) \to \mathsf{ar}(f) \\ a &: \mathsf{ar}(g) \to \mathbb{L} \end{aligned}$$

$$((\sigma', a') \star (\sigma, a))(i) := (\sigma(\sigma'(i)), a'_i \star_{\mathbb{L}} a_{\sigma'(i)})$$

Interpretation in $\mathcal{D}_f := D^{\mathsf{ar}(f)}$

Let $(\sigma, a) \in \mathcal{L}(f, g)$.

$$v \, [\![\sigma, a]\!] \, w \;:\Longleftrightarrow\; v \, [\![a]\!] \, w \circ \sigma \;:\Longleftrightarrow\; \forall i \in \mathsf{ar}(g).\ v_i \, [\![a_i]\!] \, w_{\sigma(i)}$$

Slide 12

Good Call Graph

For each $f \in \mathcal{F}$ a permutation

$$\pi_f : \mathsf{ar}(f) \to \mathsf{ar}(f)$$

s.th. for each cycle $Z := f \stackrel{(\sigma,a)}{\longrightarrow}^+ f$ there is a $1 \leq k(Z) \leq \mathsf{ar}(f)$ fulfilling

$$\sigma \circ \pi_f \restriction k(Z) = \pi_f \restriction k(Z) \qquad (1)$$

$$a \circ \pi_f \restriction k(Z) = <^{k(Z)} \qquad (2)$$

where

$$\begin{aligned} <^k &:= (\leq, \ldots, \leq, <) \in \mathbb{L}^k \\ \leq^k &:= (\leq, \ldots, \leq, \leq) \in \mathbb{L}^k \end{aligned}$$

Slide 13

Function Ordering $\lhd$

$$g \lhd f \quad :\Longleftrightarrow \quad \exists f \xrightarrow{(\sigma,a)} g.\ \forall Z.\ Z = h \longrightarrow^* f \xrightarrow{(\sigma,a)} g \xrightarrow{\tau}{}^* h$$
$$\Rightarrow\ a \circ \tau \circ \pi_h \restriction k(Z) = \leq^{k(Z)}$$

Lemma. The relation $\lhd^+ \subseteq \mathcal{F} \times \mathcal{F}$ is irreflexive.

Let $f \xrightarrow{\sigma} g$.

$$v <_h w \quad :\Longleftrightarrow \quad \forall Z = h \longrightarrow^* f \xrightarrow{\sigma} g \xrightarrow{\tau}{}^* f.$$
$$v \circ \tau \circ \pi_h \restriction k(Z)\ [\![<^{k(Z)}]\!]\ w \circ \sigma \circ \tau \circ \pi_h \restriction k(Z)$$
$$v \leq_h w \quad :\Longleftrightarrow \quad \forall Z = h \longrightarrow^* f \xrightarrow{\sigma} g \xrightarrow{\tau}{}^* f.$$
$$v \circ \tau \circ \pi_h \restriction k(Z)\ [\![\leq^{k(Z)}]\!]\ w \circ \sigma \circ \tau \circ \pi_h \restriction k(Z)$$

Evaluation Ordering $\ll$

Slide 14

$$(g, v) \ll (f, w) \ :\Longleftrightarrow\ g \prec f \vee (g \approx f \wedge \exists f \xrightarrow{(\sigma,a)} g.$$
$$\forall h \approx f.\ v \leq_h w \wedge\ (\exists h \approx f.\ v <_h w \vee\ g \lhd f))$$

Theorem. $\ll$ is a wellfounded evaluation ordering.