On η-Expansion in NbE and Type Casts

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This small note justifies the asymmetry in the definition of the η-expansion functions $\uparrow$ and $\downarrow$ for function types in the context of normalization by evaluation for dependent types [ACD07, Abe10, ACP11].

$$\begin{array}{rcl} (\uparrow^{\mathsf{Fun}\,A\,F}\,n)\,a & = & \uparrow^{F\,a}\,n\,(\downarrow^{A}\,a) \\ (\downarrow^{\mathsf{Fun}\,A\,F}\,f)\,x & = & \downarrow^{F\,\uparrow^{A}\,x}\,f\,(\uparrow^{A}\,x) \end{array}$$

The asymmetry $F\,a$ vs. $F\,\uparrow^{A}x$ can be derived from a view of $\uparrow$ and $\downarrow$ as embedding-projection pair, or up- and downcast functions, with a symmetric definition:

$$\begin{array}{rcl} (\uparrow^{\mathsf{Fun}\,A'\,F'}_{\mathsf{Fun}\,A\,F}\,f)\,a' & = & \uparrow^{F'\,a'}_{F\,(\downarrow^{A'}_{A}\,a)}\,f\,(\downarrow^{A'}_{A}\,a') \\ (\downarrow^{\mathsf{Fun}\,A'\,F'}_{\mathsf{Fun}\,A\,F}\,f')\,a & = & \downarrow^{F'\,(\uparrow^{A'}_{A}\,a)}_{F\,a}\,f\,(\uparrow^{A'}_{A}\,a) \end{array}$$

Upcast $\uparrow^{A'}_{A}$ casts a value of type A to one of type A'; downcast $\downarrow^{A'}_{A}$ casts in the opposite direction, from A' to A. The definition is symmetric, we have for all A, A',

$$\uparrow^{A'}_{A} = \downarrow^{A}_{A'}$$

The upcast $\uparrow^{A}n$ in the context of NbE casts a neutral $n \in \mathsf{Ne}$ into A, and the downcast $\downarrow^{A}a$ casts a value $a \in A$ into the set of normal forms Nf. Since

$$\mathsf{Ne} \subseteq \mathsf{Nf} \to \mathsf{Ne} \text{ and } \mathsf{Ne} \to \mathsf{Nf} \subseteq \mathsf{Nf}$$

we can identify Ne with $\mathsf{Fun}\,\mathsf{Nf}\,\lambda\_\mathsf{Ne}$ and Nf with $\mathsf{Fun}\,\mathsf{Ne}\,\lambda\_\mathsf{Nf}$. More precisely, if we want to convert from Ne it is sufficient to convert from $\mathsf{Nf} \to \mathsf{Ne}$. And if we want to convert into Nf it is sufficient to convert into $\mathsf{Ne} \to \mathsf{Nf}$. This yields

$$\begin{array}{rcccl} (\uparrow^{\mathsf{Fun}\,A'\,F'}_{\mathsf{Ne}}\,f)\,a' & = & (\uparrow^{\mathsf{Fun}\,A'\,F'}_{\mathsf{Fun}\,\mathsf{Nf}\,\lambda\_\mathsf{Ne}}\,f)\,a' & = & \uparrow^{F'\,a'}_{\mathsf{Ne}}\,f\,(\downarrow^{A'}_{\mathsf{Nf}}\,a') \\ (\downarrow^{\mathsf{Fun}\,A'\,F'}_{\mathsf{Nf}}\,f')\,a & = & (\downarrow^{\mathsf{Fun}\,A'\,F'}_{\mathsf{Fun}\,\mathsf{Ne}\,\lambda\_\mathsf{Nf}}\,f')\,a & = & \downarrow^{F'\,(\uparrow^{A'}_{\mathsf{Ne}}\,a)}_{\mathsf{Nf}}\,f\,(\uparrow^{A'}_{\mathsf{Ne}}\,a) \end{array}$$

from which we can drop the redundant indices Ne and Nf to arrive at the NbE-definition of $\uparrow$ and $\downarrow$.

References

[Abe10] Andreas Abel. Towards Normalization by Evaluation for the $\beta\eta$-Calculus of Constructions. In *FLOPS'10*, volume 6009 of *LNCS*, pages 224–239. Springer, 2010.

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[ACP11] Andreas Abel, Thierry Coquand, and Miguel Pagano. A modular type-checking algorithm for type theory with singleton types and proof irrelevance. *LMCS*, 7(2:4):1–57, 2011.