Normalization by Evaluation in the Delay Monad:
An Extended Case Study for Coinduction
Via Copatterns and Sized Types

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Abstract

In this paper, we present an Agda formalization of a normalizer for simply-typed lambda terms and its accompanying normalization proof. The normalizer consists of two coinductively defined functions in the delay monad: one is a standard evaluator of lambda terms to closures, the other a type-directed reifier from values to eta-long beta-normal forms. Their composition, normalization-by-evaluation, is shown to be a total function a posteriori, using a standard logical-relations argument. This paper builds on our workshop paper [Abel and Chapman, 2014] which presented only the normalizer and termination proof. Here we also show soundness and completeness of the normalizer, thus completing the normalization proof. The complete formalization serves as a proof-of-concept for coinductive programming and reasoning using sized types and copatterns, a new and presently experimental feature of Agda.

1 Introduction

In dependently typed programming languages such as Agda [AgdaTeam, 2014], recursively defined functions must be \textit{structurally} recursive. This requirement has the effect of fusing the program and its termination proof. It is very convenient if the function is naturally structurally recursive as there is no extra proof effort. However, it is inconvenient when the structure on which the function is structurally recursive is not visible or its comprehension is beyond the capabilities of the termination checker. Whilst there is nothing wrong with writing algorithms that are \textit{a priori} structurally recursive (indeed, it should be encouraged), we would also like to support a more step-by-step process of development where the user can write an program that is not structural and prove termination later. Yet languages based on the Curry-Howard-Isomorphism, such as Coq [INRIA, 2012] and Agda, forbid definitions that do not denote total functions, as they can be exploited to prove false theorems. However, rather than writing a \textit{terminating} recursive function in the first instance, one can write a \textit{productive} corecursive function. A potential recursive function $f : A \rightarrow B$ that would be rejected by the termination checker can be written as a corecursive function $f : A \rightarrow \text{Delay}
B$, where the type $\text{Delay}\,B$ represents values that can arrive after a possibly infinite delay [Capretta, 2005]. Showing termination amounts to proving that such a delay is in fact only finite: $\exists\,b.\,f\,a\Downarrow b$. By combining the corecursive function and its termination proof one gets a terminating recursive function. In this paper we demonstrate this approach on an canonical normalization-by-evaluation (NBE) style normalizer for simply typed lambda calculus that is not structurally recursive. Its main difference from standard NBE is that it uses first-order closures as semantic values.

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2 Delay Monad

The type \(\text{Delay} A\) of possibly non-terminating computations of type \(A\) is the greatest fixed-point of the functor \(F(X) = A + X\). In Agda, the \(\text{Delay}\) type can be represented as a mutual definition of an inductive datatype and a coinductive record. The record \(\infty\text{Delay}\) is a coalgebra and one interacts with it by using its single observation (copattern) \text{force}. Once forced we get an element of the \(\text{Delay}\) datatype which we can pattern match on to see if the value is available now or later. If it is later, then we get an element of \(\infty\text{Delay}\) which we can force again, and so forth. In Agda syntax, this is expressed as follows:

\[
\begin{align*}
\text{mutual data Delay } (A : \text{Set}) : \text{Set} \text{ where} \\
\text{now} &: A \rightarrow \text{Delay} A \\
\text{later} &: \infty\text{Delay} A \rightarrow \text{Delay} A
\end{align*}
\]

\[
\begin{align*}
\text{record } \infty\text{Delay } (A : \text{Set}) : \text{Set} \text{ where} \\
\text{coinductive field force} &: \text{Delay} A
\end{align*}
\]

We define convergence \(a \Downarrow\ a\) as a relation between delayed computations of type \(\text{Delay} A\) and values of type \(A\). If \(a \Downarrow\ a\), then the delayed computation \(a\) eventually yields the value \(a\). This is a central concept in this paper, as we will write a (productive) normalizer that produces delayed normal forms and then prove that all such delayed normal forms converge to a value, yielding termination of the normalizer. Notice that convergence is an inductive relation defined on coinductive data.

\[
\begin{align*}
\text{data } \Downarrow (A : \text{Set}) : (a? : \text{Delay} A) (a : A) \rightarrow \text{Set} \text{ where} \\
\text{now}\Downarrow &: \forall (a) \rightarrow \text{now} a \Downarrow a \\
\text{later}\Downarrow &: \forall (a) (a\infty : \infty\text{Delay} A) \rightarrow \text{force} a\infty \Downarrow a \rightarrow \text{later} a\infty \Downarrow a
\end{align*}
\]

3 Normalization

We construct a normalization function \(\text{nf}\) that takes a well-typed term \(t\) and returns a potentially delayed normal form.

\[\text{nf} : \forall (\Gamma a) (t : \text{Tm} \Gamma a) \rightarrow \text{Delay} (\text{Nf} \Gamma a)\]

The termination proof states that for any term \(t\), there exists a normal form \(n\) such that the result of normalization \(\text{nf}\) converges to \(n\) after a finite delay.

\[\text{normalize} : \forall (\Gamma a) (t : \text{Tm} \Gamma a) \rightarrow \exists \lambda n \rightarrow \text{nf} t \Downarrow n\]

Soundness of the normalizer states that any two terms \(t\) and \(t'\) that are related in the equational theory \(t \sim t'\) have equal normal forms. The normal form \(n\) is obtained as the existential witness (first projection \(\text{fst}\)) of a \text{run} of the normalization proof \text{normalize} \(t\).

\[\text{sound} : \forall (\Gamma a) (t \infty \Gamma) (t : \text{Tm} \Gamma a) \rightarrow t \sim t' \rightarrow \text{fst} (\text{normalize} t) \equiv \text{fst} (\text{normalize} t')\]

The main work in the completeness proof is to conclude that any term \(t\) is related in the equational theory to its normal form.

\[\text{complete-lemma} : \forall (\Gamma a) (t \infty \Gamma) (t : \text{Tm} \Gamma a) \rightarrow t \sim \text{emb} (\text{fst} (\text{normalize} t))\]

Having proved \text{soundness} and \text{complete-lemma}, completeness follows as a corollary.

\[\text{complete} : \forall (\Gamma a) (t \infty \Gamma) (t : \text{Tm} \Gamma a) \rightarrow \text{fst} (\text{normalize} t) \equiv \text{fst} (\text{normalize} t') \rightarrow t \sim t'\]
4 Conclusion

Our case study suggests that encoding partial functions via the delay monad is a viable alternative to encoding via inductive functional relations even for non-trivial functions such as an evaluator. In future work, we would like to investigate whether the technique is beneficial in the daunting task of a formalized semantics of dependent type theory [Chapman 2009].

References


