

Normalization by Evaluation for Sized Dependent Types

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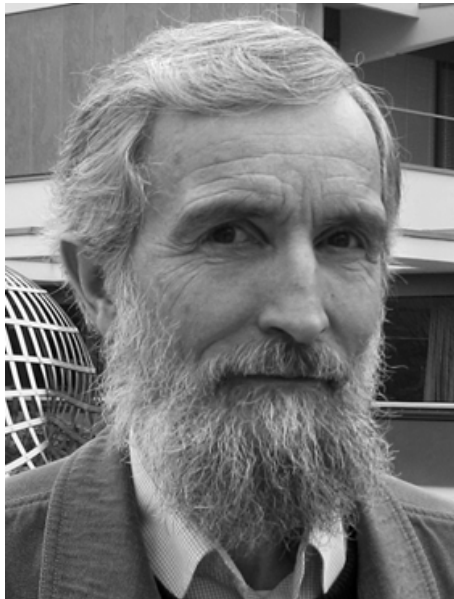
Dependent type theory à la Martin-Löf

- Typing judgement $\Gamma \vdash t : A$
 - in context Γ , expression t has type A
 - under hypotheses Γ , term t is a proof of proposition A
- Consistency: empty type not inhabited $\not\vdash t : \perp$.
- Conversion / subsumption rule

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash A \leq B}{\Gamma \vdash t : B}$$

- Definitional equality $\Gamma \vdash t = t' : A$
in Γ , terms t and t' of type A are not distinguished
- Reflexivity rule of subtyping

$$\frac{\Gamma \vdash A = B : \text{Type}}{\Gamma \vdash A \leq B}$$



Subtyping??

Definitional equality

- Decidable (for type checking).
- Proper subrelation of extensional equality (usually $\not\vdash x + y = y + x$).
- Equates at least computationally equal terms.
- The bigger, the better!
- Type needed for η -laws, e.g., for unit type \top .

$$\frac{\Gamma \vdash t : T \quad \Gamma \vdash t' : T}{\Gamma \vdash t = t' : T}$$

Sized types as refinement types

- Termination needed for consistency.
 $\text{fix} : ((\text{Nat} \rightarrow C) \rightarrow (\text{Nat} \rightarrow C)) \rightarrow \text{Nat} \rightarrow C$ inconsistent
- Index data types by tree height: “ $\text{Nat}^i = \{n \mid n < i\}$ ”
 $\text{fix} : (\forall i \rightarrow (\text{Nat}^i \rightarrow C) \rightarrow (\text{Nat}^{i+1} \rightarrow C)) \rightarrow \forall i \rightarrow \text{Nat}^i \rightarrow C$
- Polymorphism $t : (\forall i \rightarrow \text{Tree}^i \rightarrow \text{Tree}^i) \rightarrow \forall i \rightarrow \text{Tree}^i \rightarrow \text{Tree}^\infty$
- Subtyping: $\text{Nat}^0 \leq \text{Nat}^1 \leq \text{Nat}^2 \leq \dots \leq \text{Nat}^\infty$
- Size expressions are not unique. Problematic for proofs.

$$\begin{aligned} \text{suc} & : \forall i \rightarrow \text{Nat}^i \rightarrow \text{Nat}^{i+1} \\ \text{suc } i & : \text{Nat}^i \rightarrow \text{Nat}^\infty \\ \text{suc } (i + 1) & : \text{Nat}^i \rightarrow \text{Nat}^\infty \\ \text{suc } \infty & : \text{Nat}^i \rightarrow \text{Nat}^\infty \end{aligned}$$

- Intuition: Sizes are *irrelevant* in terms, but *relevant* in types.

Grau ist alle Theorie

Agda demo

Church-style vs. Curry style

- Pure Type Systems: Church-style dependent functions = polymorphic functions.

$$\frac{\Gamma \vdash t = t' : (i : \text{Size}) \rightarrow A \quad \Gamma \vdash b : \text{Size}}{\Gamma \vdash t b = t' b : A[b/i]}$$

- Curry-style would fit our needs:

$$\frac{\Gamma \vdash t = t' : \forall i : \text{Size}. A \quad \Gamma \vdash b : \text{Size}}{\Gamma \vdash t = t' : A[b/i]}$$

- Absent size arguments cannot spoil our proofs.
- Synthesis: *Churry-style*.

Church-style vs. Curry style

- Pure Type Systems: Church-style dependent functions = polymorphic functions.

$$\frac{\Gamma \vdash t = t' : (i : \text{Size}) \rightarrow A \quad \Gamma \vdash b : \text{Size}}{\Gamma \vdash t b = t' b : A[b/i]}$$

- Curry-style would fit our needs:

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- Absent size arguments cannot spoil our proofs.
- Synthesis: *Churry-style*.

Kind of Churry



“Churry-style”

- Use size arguments for *type checking*.
- Ignore them during *equality checking*.
- Irrelevant size quantification:

$$\frac{\Gamma \vdash t = t' : (\bullet i : \text{Size}) \rightarrow A \quad \Gamma \vdash a, a', b : \text{Size}}{\Gamma \vdash t \langle a \rangle = t' \langle a' \rangle : A[b/i]}$$

- Forces to ignore arguments also during *typing*.

$$\frac{\Gamma \vdash t : (\bullet i : \text{Size}) \rightarrow A \quad \Gamma \vdash a, b : \text{Size}}{\Gamma \vdash t \langle a \rangle : A[b/i]}$$

- (Bidirectional) *type checking*:

$$\frac{\Gamma \vdash t \Rightarrow (\bullet i : \text{Size}) \rightarrow A \quad \Gamma \vdash a \Leftarrow \text{Size}}{\Gamma \vdash t \langle a \rangle \Rightarrow A[a/i]}$$

Does “Churry-style” work in general?

- Generally, can we have?

$$\frac{\Gamma \vdash t = t' : (\bullet x : A) \rightarrow B \quad \Gamma \vdash u, u', v : A}{\Gamma \vdash t \langle u \rangle = t' \langle u' \rangle : B[v/x]}$$

- Not with type-directed η !

$$\frac{\vdash f : (\bullet X : \text{Type}) \rightarrow (X \rightarrow X) \rightarrow C}{\begin{aligned} &\vdash f \langle A \rightarrow A \rangle (\lambda xy \rightarrow xy) \\ &= f \langle A \rightarrow \top \rangle (\lambda xy \rightarrow _) \\ &= f \langle A \rightarrow A \rangle (\lambda xy \rightarrow x(xy)) \end{aligned}}$$

- In $(\bullet x : A) \rightarrow B$, the *shape* of B must be independent of x .
 - Same shape: Nat^i , Nat^{j+1} , Nat^∞ .
 - Different shape: \top , \perp , $A \rightarrow B$.

Shape-irrelevance

- These are shape-irrelevant in i
 - $i \vdash \text{Nat}^i$
 - $i \vdash \text{Nat}^i \rightarrow \text{Nat}^{i+1}$
 - $i \vdash \text{if } b \text{ then } \text{Nat}^i \text{ else } \text{Nat}^i \rightarrow \text{Nat}^i$
 - $i \vdash (x : \text{Nat}^i) \rightarrow \text{Vec } A \ x$

- These are *not* shape-irrelevant:
 - $b : \text{Bool} \not\vdash \text{if } b \text{ then } \text{Type} \text{ else } \text{Type} \rightarrow \text{Type}$
 - $X : \text{Type} \not\vdash X$
 - $X : \text{Type} \not\vdash X \rightarrow X$

Official rules for irrelevant quantifier

- Formation:

$$\frac{\Gamma \vdash A : \text{Type} \quad \Gamma, \bullet\bullet x : A \vdash B : \text{Type}}{\Gamma \vdash (\bullet x : A) \rightarrow B : \text{Type}}$$

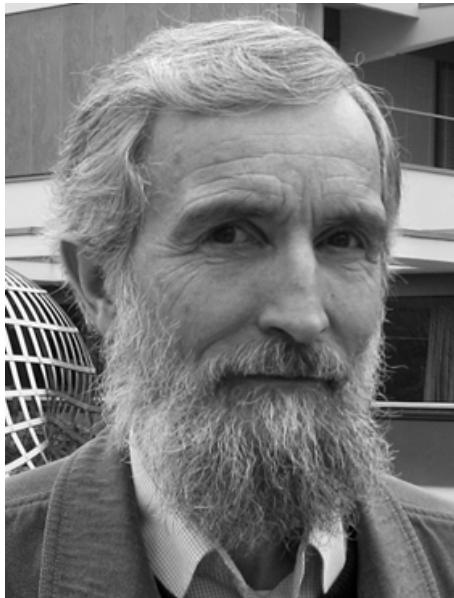
- Introduction:

$$\frac{\Gamma, \bullet x : A \vdash t : B}{\Gamma \vdash (\lambda x \rightarrow t) : (\bullet x : A) \rightarrow B}$$

- Elimination:

$$\frac{\Gamma \vdash t : (\bullet x : A) \rightarrow B \quad \bullet^{-1}(\Gamma) \vdash u, v : A}{\Gamma \vdash t u : B[v/x]}$$

- Resurrection* $\bullet^{-1}(\Gamma)$ removes \bullet s. (Nuyts et al.: $\bullet \setminus \Gamma$)



Irrelevance??

In the paper

- Dependent type theory with
 - predicative universes $\text{Set}_0, \text{Set}_1, \dots$,
 - sized natural numbers Nat^i ,
 - subtyping
 - irrelevant \forall and shape-irrelevant Π size quantifier.
- Decidability of definitional equality via normalization-by-evaluation (NbE).
- Decidability of bidirectional type checking.
- Discussion of related irrelevant quantifiers (Pfenning, Miquel, Barras/Bernardo, Mishram-Linger/Sheard).

Normalization by Evaluation (NbE)

- Type-directed eta-expansion:
 - Reflection $\uparrow^A t$: syntax to semantics.
 - Reification $\downarrow^A d$: semantics to syntax.
- My observation: only *shape* of A matters.
- Works for *type assignment* NbE
[e.g., Abel/Aehlig/Coquand/Dybjer, 2007–].
- Not for *intrinsically typed* NbE
[e.g., Altenkirch/Kaposi, FSCD 2016].

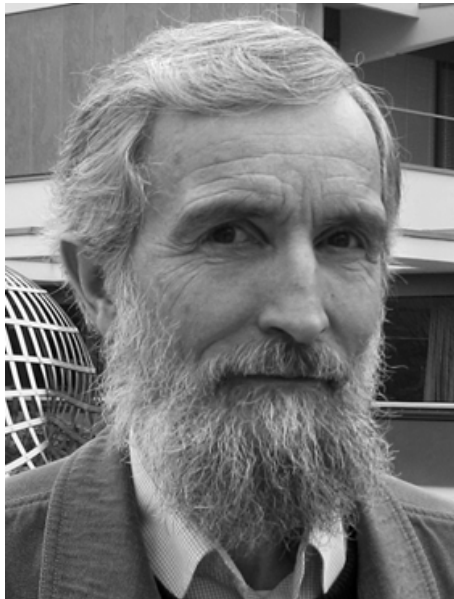
Take home message

The irrelevance modality allows us to ignore sizes where they just help the type checker.

$\text{suc}\langle\text{ignoreMe}\rangle n : \text{Nat}^{\text{dontIgnoreMe}}$

Future work

- Explore *shape-irrelevance*.
- Replace NbE by traditional algorithmic equality.
- Solidify Agda implementation of shape-irrelevance.



Sized types?!