Strong Normalization for Guarded Recursive Types

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Theory Seminar
Institute of Cybernetics, Tallinn, Estonia
18 December 2014
Introduction

- Guarded recursive types (Nakano, LICS 2000)
- Negative recursive types while maintaining consistency
  - $\mu X. ▶X \rightarrow A$
  - fix : ($\uparrow A \rightarrow A$) $\rightarrow A$

Applications
- Semantics (abstracting step-indexing)
- Functional Reactive Programming (causality)
- Coinduction (productivity, with a “Globally”/“□” modality)

This talk: Strong Normalization.
Guarded types

- Types and terms.

\[ A, B ::= A \rightarrow B \mid ▶A \mid X \mid \mu X. A \]
\[ t, u ::= x \mid \lambda x. t \mid t u \mid \text{next } t \mid t \ast u \]

- Occurrences of \( X \) in \( \mu X. A \) must be under a ▶ “guard”.

**Good:**
- \( \mu X. ▶X \)
- \( \mu X. A \times ▶X \) and \( \mu X. ▶(A \times X) \)
- \( \mu X. (▶X) \rightarrow A \) and \( \mu X. ▶(X \rightarrow A) \).

**Bad:**
- \( \mu X. X \) and \( \mu X. A \times X \)
- \( \mu X. X \rightarrow A \) and \( \mu X. X \rightarrow ▶A \)
- \( \mu X. ▶\mu X. X \).
Typing

- Type equality: congruence closure of $\vdash \mu X. A = A[\mu X. A/X]$.
- Typing $\Gamma \vdash t : A$.

\[
\begin{align*}
\Gamma \vdash t : A & \quad \Gamma \vdash t : \triangleright (A \rightarrow B) & \quad \Gamma \vdash u : \triangleright A \\
& \quad \Gamma \vdash t \ast u : \triangleright B \\
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash t : A & \quad \vdash A = B \\
& \quad \Gamma \vdash t : B
\end{align*}
\]
Denotational Semantics

Types as streams of sets:
\( A : \mathbb{N} \rightarrow \text{Set} \) with restriction maps.
**Fixed-point construction (intuition)**

Any map $f : \textit{A} \to \textit{A}$ has a fixed-point $\textit{fix}_f : 1 \to \textit{A}$:
Reduction

- Redex contraction $t \mapsto t'$.

$$
(\lambda x. t) u \mapsto t[u/x] \\
\text{next } t \ast \text{next } u \mapsto \text{next } (t u)
$$

- Full one-step reduction $t \longrightarrow t'$: Compatible closure of $\mapsto$.
Recursion from recursive types

Guarded recursion combinator can be encoded.
The standard $Y$ combinator would need a type $T$ such that

$$T = T \rightarrow A$$

to typecheck the self applications of $x$ and $\omega$:

$$f : A \rightarrow A$$

$$xx : A \quad \text{if} \ x : T$$

$$\omega := \lambda (x : T). f\ (xx) : T \rightarrow A$$

$$Y := \omega \omega : A$$
Recursion from recursive types

We can solve $T = \top A$:

$$T = \mu X. \top X \rightarrow A$$

So we get a guarded fixpoint combinator:

$$
\begin{align*}
 f & : \top A \rightarrow A \\
 x & : \top (\top T \rightarrow A) \text{ if } x : \top T \\
 x \ast \text{next } x & : \top A \text{ if } x : \top T \\
 \omega & := \lambda (x : \top T). f (x \ast \text{next } x) : \top T \rightarrow A \\
 Y_f & := \omega (\text{next } \omega) : A
\end{align*}
$$

$$Y_f \xrightarrow{} f (\text{next } \omega \ast \text{next } (\text{next } \omega)) \xrightarrow{} f (\text{next } (\omega (\text{next } \omega))) = f (\text{next } Y_f)$$

Note: Full reduction $\rightarrow$ of $Y_f$ diverges.
More Examples

- Streams!? 
- RepMin: One pass through binary tree, replacing all labels by their minimum. 
- Attribute grammars!? 
Restricted reduction

- Restore normalization: do not reduce under `next`.
- Relaxed: reduce only under `next` up to a certain depth.
- Family $\rightarrow_n$ of reduction relations.

\[
\begin{align*}
t &\mapsto t' \\
t &\rightarrow_n t' \\
t &\rightarrow_n t' \\
\text{next } t &\rightarrow_{n+1} \text{next } t'
\end{align*}
\]

- Plus compatibility rules for all other term constructors.
- $\rightarrow_n$ is monotone in $n$ (more fuel gets you further).
- Goal: each $\rightarrow_n$ is strongly normalizing.
Restricted reduction (Example)

\[ Y \xrightarrow{\ast}^0 f(\text{next } Y) \xrightarrow{\cdot}^0 \]

\[ Y \xrightarrow{\ast}^1 f(\text{next}(f(\text{next } Y))) \xrightarrow{\cdot}^1 \]

\[ Y \xrightarrow{\ast}^2 f(\text{next}(f(\text{next}(f(\text{next } Y))))) \xrightarrow{\cdot}^2 \]

\[ \vdots \]
Strong normalization as well-foundedness

- \( t \in sn_n \) if \( \rightarrow_n \) reduction starting with \( t \) terminates.

\[ \forall t'. t \rightarrow_n t' \implies t' \in sn_n \]

- \( sn_n \) is antitone in \( n \), since \( \rightarrow_n \) occurs negatively.
- More reductions \( \implies \) less termination.
Inductive $\text{SN}_n$

- Take the inductively defined normal forms:

\[
E ::= \_ | E \ u | E \ast u | \text{next } t \ast E
\]

\[
\begin{align*}
E \in \text{SN}_n & \quad \Rightarrow \quad E[x] \in \text{SN}_n \\
\lambda x. t \in \text{SN}_n & \quad \Rightarrow \quad \text{next } t \in \text{SN}_0 \\
\text{next } t \in \text{SN}_n & \quad \Rightarrow \quad \text{next } t \in \text{SN}_{n+1}
\end{align*}
\]

- And close them under “Strong head reduction” $t \xrightarrow{\text{SN}} t'$

\[
\begin{align*}
t \xrightarrow{\text{SN}} t' & \quad \Rightarrow \quad t' \in \text{SN}_n \\
E[t] \xrightarrow{\text{SN}} E[t'] & \quad \Rightarrow \quad t \xrightarrow{n} t'
\end{align*}
\]

- $t \xrightarrow{\text{SN}} t'$ is like weak head reduction but erased terms must be s.n.
Notions of s.n. coincide?

- Rules for $SN_n$ are closure properties of $sn_n$.
- $SN_n \subseteq sn_n$ follows by induction on $SN_n$.
- Converse $sn_n \subseteq SN_n$ does not hold!
- Counterexamples are ill-typed s.n. terms, e.g.,
  \[(\lambda x. x)^* y\] or \[(\text{next } x) y\].

Solution: consider only well-typed terms.

Proof of $t \in sn_n \implies t \in SN_n$ by case distinction on $t$: neutral ($E[x]$), introduction ($\lambda x.t$, $\text{next } t$), or weak head redex.
Saturated sets (semantic types)

- Types are modeled by sets $\mathcal{A} \subseteq \text{SN}_n$.
- $n$-closure $\overline{\mathcal{A}}_n$ of $\mathcal{A}$ inductively:

  \[
  \begin{align*}
  t \in \mathcal{A} & \implies t \in \overline{\mathcal{A}}_n \\
  E \in \text{SN}_n & \implies E[x] \in \overline{\mathcal{A}}_n \\
  t \xrightarrow{\text{SN}_n} t' & \implies t' \in \overline{\mathcal{A}}_n
  \end{align*}
  \]

- $\mathcal{A}$ is $n$-saturated ($\mathcal{A} \in \text{SAT}_n$) if $\overline{\mathcal{A}}_n \subseteq \mathcal{A}$.
- Saturated sets are non-empty (contain e.g. the variables).
Constructions on semantic types

- Function space and “later”:

\[
\mathcal{A} \rightarrow \mathcal{B} = \{ t \mid t \ u \in \mathcal{B} \text{ for all } u \in \mathcal{A} \}
\]

\[
\triangleright_n \mathcal{A} = \{ \text{next } t \mid t \in \mathcal{A} \text{ if } n > 0 \}_n
\]

- If \( \mathcal{A}, \mathcal{B} \in \text{SAT}_n \) then \( \mathcal{A} \rightarrow \mathcal{B} \in \text{SAT}_n \).
- \( \triangleright_0 \mathcal{A} \in \text{SAT}_0 \).
- If \( \mathcal{A} \in \text{SAT}_n \) then \( \triangleright_{n+1} \mathcal{A} \in \text{SAT}_{n+1} \).
Type interpretation

- Type interpretation $\llbracket A \rrbracket_n \in \text{SAT}_n$

$$\llbracket A \rightarrow B \rrbracket_n = \bigcap_{n' \leq n} (\llbracket A \rrbracket_{n'} \rightarrow \llbracket B \rrbracket_{n'})$$

$$\llbracket A \rrbracket_0 = \uparrow_0 \text{SN}_0 = \{ \text{next } t \}_0$$

$$\llbracket A \rrbracket_{n+1} = \uparrow_{n+1} \llbracket A \rrbracket_n$$

$$\llbracket \mu X. A \rrbracket_n = \llbracket A[\mu X. A/X] \rrbracket_n$$

- By lex. induction on $(n, \text{size}(A))$ where $\text{size}(\uparrow A) = 0$.

- Requires recursive occurrences of $X$ to be guarded by a $\uparrow$. 
Type soundness

- Context interpretation:
  \[ \rho \in \llbracket \Gamma \rrbracket_n \iff \rho(x) \in \llbracket A \rrbracket_n \text{ for all } (x:A) \in \Gamma \]

- Identity substitution: \( \text{id} \in \llbracket \Gamma \rrbracket_n \) since \( x \in \llbracket A \rrbracket_n \).

- Type soundness: if \( \Gamma \vdash t : A \) then \( t\rho \in \llbracket A \rrbracket_n \) for all \( n \) and \( \rho \in \llbracket \Gamma \rrbracket_n \).

- Corollary: \( t \in SN_n \) for all \( n \).
Formalization in Agda

Syntax of types as a mixed inductive-coinductive datatype:

\[ Ty = \nu X \mu Y. (Y \times Y) + X \]

mutual
data Ty : Set where
  _→_ : (a b : Ty) → Ty
  ▼_ : (a∞ : ∞Ty) → Ty

record ∞Ty : Set where
  coinductive
  constructor delay_
  field force_ : Ty

- Intensional (propositional) equality too weak for coinductive types.
- \(\xrightarrow{\text{add an extensionality axiom for our coinductive type.}}\)
**Well-typed terms**

```haskell
data Tm (Γ : Cxt) : (a : Ty) → Set where
  var : ∀{a} (x : Var Γ a) → Tm Γ a
  abs : ∀{a b} (t : Tm (a :: Γ) b) → Tm Γ (a → b)
  app : ∀{a b} (t : Tm Γ (a → b)) (u : Tm Γ a) → Tm Γ b
  next : ∀{a∞} (t : Tm Γ (force a∞)) → Tm Γ (▷ a∞)
  _*_* : ∀{a∞ b∞} (t : Tm Γ (▷ (a∞ ⇒ b∞))) (u : Tm Γ (▷ a∞)) → Tm Γ (▷ b∞)
```

- We used intrinsically well-typed terms (data structure indexed by typing context and type expression).
- Second Kripke dimension (context) required “everywhere”, e.g., in SN and [A].
Conclusions & Further work

- **Strong** normalization is a new result, albeit expected for the restricted reduction.
- Agda formalization (ca. 3kLoc, 170kB) useful as basis for further research.
- Add modalities to handle (co)inductive types.
- Integrate into Intensional Type Theory.