# Copatterns Programming Infinite Objects by Observations

#### Andreas Abel

Department of Computer Science Ludwig-Maximilians-University Munich

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## Copatterns

- Originated from AIM discussions since 2008 on coinduction.
- MiniAgda prototype (March 2011).
- Started on Agda prototype in Nov 2011 (with James Chapman)
- Currently in abstract syntax only
- Goal: integrate into the core (internal syntax)



## What's wrong with Coq's Coinductive?

Coq's coinductive types are non-wellfounded data types.

```
CoInductive U : Type :=
| inn : U -> U.
CoFixpoint u : U := inn u.
```

 Reduction of cofixpoints is context-sensitive, to maintain strong normalization.

```
case (inn s) of inn y \Rightarrow t = t[s/y]
case (cofix f) of inn y \Rightarrow t = \text{case}(f(\text{cofix } f)) of inn y \Rightarrow t
```

## A problem with subject reduction in Coq

```
: Type
                                                  a codata type
inn : U \rightarrow U
                                            its (co)constructor
u : U
                                                inhabitant of U
u = cofix inn
                                                u = inn(inn(...
force : U \rightarrow U
                                                      an identity
force = \lambda x. case x of inn y \Rightarrow inn y
eq : (x : U) \rightarrow x \equiv \text{force } x dep. elimination
eq = \lambda x. case x of inn y \Rightarrow \text{refl}
eq_u : u \equiv inn u
                                                 offending term
eq_u = eq u \longrightarrow refl
                                              \not\vdash refl : u \equiv inn \ u
```

## **Analysis**

• Problematic: dependent matching on coinductive data.

$$\frac{\Gamma \vdash u : \mathsf{U} \qquad \Gamma, \ y : \mathsf{U} \vdash t : C(\mathsf{inn} \ y)}{\Gamma \vdash \mathsf{case} \ u \ \mathsf{of} \ \mathsf{inn} \ y \Rightarrow t : C(u)}$$

- Solution: Paradigm shift.
  - Understand coinduction not through construction, but through observations.
- Hinderance: The human mind seems to prefer concrete constructions over abstract black boxes with an ascribed behavior.



## Function Definition by Observation

- A function is a black box. We can apply it to an argument (experiment), and observe its result (behavior).
- Application is the defining principle of functions [Granström's dissertation 2009].

$$\frac{f:A\to B \qquad a:A}{f:a:B}$$

- $\lambda$ -abstraction is derived, secondary to application.
- Transfer this to other infinite objects: coinductive things.

## Infinite Objects Defined by Observation

- A coinductive object is a black box.
- There is a finite set of experiments (projections) we can conduct on it.
- The object is determined by the observations we make on it.
- Generalize records to coinductive types.
- Agda code:

```
record U : Set where
  coinductive
  field
  out : U
```

- out : U -> U is the experiment we can make on U.
- Objects of type U are defined by the result of this experiment.

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## Infinite Objects Defined by Observation

Defining a cofixpoint.

```
u : U
out u = u
```

• Defining the "constructor".

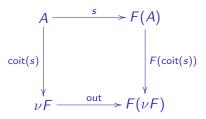
```
inn : U -> U
out (inn x) = x
```

- We call (out \_) a projection copattern.
- And (\_ x) an application copattern.
- The whole thing (out (\_ x)) is a composite copattern.



## Category-theoretic Perspective

- Functor F, coalgebra  $s: A \to F(A)$ .
- Terminal coalgebra out :  $\nu F \to F(\nu F)$  (elimination).
- Coiteration  $coit(s): A \rightarrow \nu F$  constructs infinite objects.



• Computation rule: Only unfold infinite object in elimination context.

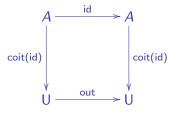
$$\operatorname{out}(\operatorname{coit}(s)(a)) = F(\operatorname{coit}(s))(s(a))$$

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Agda

#### Instance: U

- With F(X) = X we get the coinductive unit type  $U = \nu F$ .
- With  $s = id_A$  we get u = coit(id)(a) for arbitrary a : A.

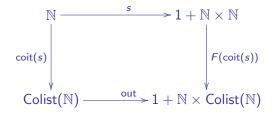


• Computation out(u) = u.

Andreas Abel (LMU)

#### Instance: Colists of Natural Numbers

- With  $F(X) = 1 + \mathbb{N} \times X$  we get  $\nu F = \text{Colist}(\mathbb{N})$ .
- With  $s(n : \mathbb{N}) = inr(n, n + 1)$  we get coit(s)(n) = (n, n + 1, n + 2, ....).





Andreas Abel (LMU)

Agda

## Colists in Agda

Colists as record.

```
data Maybe A : Set where nothing : Maybe A just : A 
ightarrow Maybe A
```

record Colist A : Set where
 coinductive
 field
 out : Maybe (A × Colist A)

• Sequence of natural numbers.

```
nats : \mathbb{N} \to \mathbb{N}
out (nats n) = just (n , nats (n + 1))
```



#### **Streams**

Streams have two observations: head and tail.

```
record Stream A : Set where
  coinductive
  field
  head : A
```

tail : Stream A

A stream is defined by its head and tail.

```
zipWith : {A B C : Set} -> (A -> B -> C) ->
   Stream A -> Stream B -> Stream C
head (zipWith f as bs) = f (head as) (head bs)
tail (zipWith f as bs) = zipWith f (tail as) (tail bs)
```

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## Deep Copatterns: Fibonacci-Stream

• Fibonacci sequence obeys this recurrence:

```
0 1 1 2 3 5 8 ... (fib)
1 1 2 3 5 8 13 ... (tail fib)
1 2 3 5 8 13 21 ... tail (tail fib)
```

This directly leads to a definition by copatterns:

```
fib : Stream \mathbb{N} (tail (tail fib)) = zipWith _+_ fib (tail fib) (head (tail fib)) = 1 ( (head fib)) = 0
```

Strongly normalizing definition of fib!



#### **Fibonacci**

Definition with cons not strongly normalizing.

```
fib = 0 :: 1 :: zipWith _+_ fib (tail fib)
```

• Diverges under Coq's reduction strategy:

## Type-theoretic motivation

Foundation: coalgebras (category theory) and focusing (polarized logic)

polarity	positive	negative
linear types	$1, \oplus, \otimes, \mu$	<b>-</b> ∘, &, ν
Agda types	data	ightarrow, record
extension	finite	infinite
introduction	constructors	definition by copatterns
elimination	pattern matching	message passing
categorical	algebra	coalgebra

## Types

$$A,B,C ::= X$$
 Type variable Positive type Negative type

 $P ::= 1$  Unit type Cartesian product Data type

 $P ::= A \rightarrow B$  Function type Record type

 $P ::= A \rightarrow B$  Function type Record type

 $P ::= C_1 A_1 | \cdots | C_n A_n$  Variant (labeled sum)

 $P ::= C_1 A_1 | \cdots | C_n A_n$  Record (labeled product)

## Type examples

• Data types (algebraic types):

```
\begin{array}{lll} \text{List } A & = & \mu X \, \langle \text{nil } 1 \mid \text{cons } (A \times X) \rangle \\ \text{Nat} & = & \mu X \, \langle \text{zero } 1 \mid \text{suc } X \rangle \\ \text{Maybe } A & = & \mu_- \langle \text{nothing } 1 \mid \text{just } A \rangle \\ 0 & = & \mu_- \langle \rangle & \text{(positive empty type)} \end{array}
```

Record types (coalgebraic types):

```
Stream A = \nu X \{ \text{head} : A, \text{tail} : X \}

Colist A = \nu X \{ \text{out} : \mu_{-} \langle \text{nil } 1 \mid \text{cons } (A \times X) \rangle \}

Vector A = \nu_{-} \{ \text{length} : \text{Nat}, \text{ elems} : \text{List } A \}

\top = \nu_{-} \{ \} (negative unit type)
```

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## **Terms**



## Bidirectional Type Checking

•  $\Delta \vdash t \Rightarrow A$  In context  $\Delta$ , the type of term t is inferred as A.

$$\frac{\Delta(x) = A}{\Delta \vdash f \Rightarrow \Sigma(f)} \qquad \frac{\Delta(x) = A}{\Delta \vdash x \Rightarrow A} \qquad \frac{\Delta \vdash t \Rightarrow \nu XR}{\Delta \vdash t.d \Rightarrow R_d[\nu XR/X]}$$
$$\frac{\Delta \vdash t_1 \Rightarrow A_1 \to A_2 \qquad \Delta \vdash t_2 \Leftarrow A_1}{\Delta \vdash t_1 \ t_2 \Rightarrow A_2}$$

•  $\Delta \vdash t \Leftarrow A$  In context  $\Delta$ , term t checks against type A.

$$\frac{\Delta \vdash t \Rightarrow A \qquad A = C}{\Delta \vdash t \Leftarrow C} \qquad \frac{\Delta \vdash t \Leftarrow D_{c}[\mu XD/X]}{\Delta \vdash c \ t \Leftarrow \mu XD}$$

$$\frac{\Delta \vdash t_{1} \Leftarrow A_{1} \qquad \Delta \vdash t_{2} \Leftarrow A_{2}}{\Delta \vdash (t_{1}, t_{2}) \Leftarrow A_{1} \times A_{2}}$$

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## Patterns and Copatterns

#### Patterns

$$\begin{array}{cccc} p & ::= & x & & \text{Variable pattern} \\ & | & () & & \text{Unit pattern} \\ & | & (p_1, p_2) & & \text{Pair pattern} \\ & | & c & p & & \text{Constructor pattern} \end{array}$$

#### Copatterns

$$egin{array}{lll} q & ::= & \cdot & & \mbox{Hole} \ & | & q & p & \mbox{Application copattern} \ & | & q.d & \mbox{Destructor copattern} \end{array}$$



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## Pattern Type Checking

•  $\Delta \vdash p \Leftarrow A$  Pattern p checks against type A, yielding  $\Delta$ .

$$\frac{\Delta \vdash p \Leftarrow D_{c}[\mu XD/X]}{\Delta \vdash c \ p \Leftarrow \mu XD}$$

$$\frac{\Delta \vdash p \Leftarrow D_{c}[\mu XD/X]}{\Delta \vdash c \ p \Leftarrow \mu XD}$$

$$\frac{\Delta_{1} \vdash p_{1} \Leftarrow A_{1} \qquad \Delta_{2} \vdash p_{2} \Leftarrow A_{2}}{\Delta_{1}, \Delta_{2} \vdash (p_{1}, p_{2}) \Leftarrow A_{1} \times A_{2}}$$

•  $\Delta \mid A \vdash q \Rightarrow C$  Copattern q eliminates given type A into inferred type C, yielding context  $\Delta$ .

$$\frac{\Delta \mid A \vdash q \Rightarrow \nu XR}{\Delta \mid A \vdash q \Rightarrow \nu AR}$$

$$\frac{\Delta \mid A \vdash q \Rightarrow R_d[\nu XR/X]}{\Delta \mid A \vdash q \Rightarrow B \rightarrow C}$$

$$\frac{\Delta_1 \mid A \vdash q \Rightarrow B \rightarrow C \qquad \Delta_2 \vdash p \Leftarrow B}{\Delta_1, \Delta_2 \mid A \vdash q \mid p \Rightarrow C}$$

## Fibonacci Example Program

- Program consists of type signatures  $\Sigma$  and rewrite rules Rules.
- Example entries for fib.

```
\begin{split} & \Sigma(\mathsf{fib}) &= \nu X \{\mathsf{head} : \mu Y \, \langle \mathsf{zero} \ 1 \, | \, \mathsf{suc} \ Y \rangle \,, \ \mathsf{tail} : X \} \\ & \mathsf{Rules}(\mathsf{fib}) = \left\{ \begin{array}{l} \cdot \, .\mathsf{head} & \mapsto \mathsf{zero} \, () \\ \cdot \, .\mathsf{tail} \, .\mathsf{head} & \mapsto \mathsf{suc} \, (\mathsf{zero} \, ()) \\ \cdot \, .\mathsf{tail} \, .\mathsf{tail} & \mapsto \mathsf{zipWith} \,\, \_+\_ \,\, \mathsf{fib} \, (\mathsf{fib} \,\, .\mathsf{tail}) \end{array} \right\} \end{split}
```



#### **Evaluation**

- Redexes have form E[f].
- Evaluation contexts E.

$$E ::= \cdot \qquad \text{Hole}$$
 $\mid E e \qquad \text{Application}$ 
 $\mid E.d \qquad \text{Projection}$ 

• To reduce redex, we need to match *E* against copatterns *q*.



# (Co)pattern Matching

•  $t = p \setminus \sigma$  Term t matches with pattern p under substitution  $\sigma$ .

$$\frac{t = \stackrel{?}{p} \searrow \sigma}{t = \stackrel{?}{x} \searrow t/x} \qquad \frac{t = \stackrel{?}{p} p \searrow \sigma}{c \ t = \stackrel{?}{c} c \ p \searrow \sigma}$$

$$\frac{t_1 = \stackrel{?}{p} p_1 \searrow \sigma_1 \qquad t_2 = \stackrel{?}{p} p_2 \searrow \sigma_2}{(t_1, t_2) = \stackrel{?}{p} (p_1, p_2) \searrow \sigma_1, \sigma_2}$$

•  $E = q \setminus \sigma$  Evaluation context E matches copattern q returning substitution  $\sigma$ .

$$\frac{E = {}^{?} q \searrow \sigma}{E.d = {}^{?} q.d \searrow \sigma}$$

$$\frac{E = {}^{?} q \searrow \sigma \qquad t = {}^{?} p \searrow \sigma'}{E t = {}^{?} q p \searrow \sigma, \sigma'}$$

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## Interactive Program Development

• Goal: cyclic stream of numbers.

```
cycleNats : \mathbb{N} \to \mathsf{Stream} \ \mathbb{N} cycleNats n = n, n-1, \ldots, 1, 0, N, N-1, \ldots, 1, 0, \ldots
```

• Fictuous interactive Agda session.

```
 \begin{array}{lll} \mathsf{cycleNats} & : & \mathsf{Nat} \to \mathsf{Stream} \ \mathsf{Nat} \\ \mathsf{cycleNats} & = & ? \end{array}
```

• Split result (function).

cycleNats 
$$x = ?$$

Split result again (stream).

```
head (cycleNats x) = ? tail (cycleNats x) = ?
```



## Interactive Program Development

Last state:

```
head (cycleNats x) = ? tail (cycleNats x) = ?
```

Split x in second clause.

```
head (cycleNats x) = ?
tail (cycleNats 0) = ?
tail (cycleNats (1 + x')) = ?
```

• Fill right hand sides.

```
head (cycleNats x) = x
tail (cycleNats 0) = cycleNats N
tail (cycleNats (1 + x')) = cycleNats x'
```



## Copattern Coverage

- Coverage algorithm:
- Start with the trivial covering (copattern · "hole").
- Repeat
  - split result or
  - split a pattern variable

until computed covering matches user-given patterns.

## Coverage Rules

 $A \triangleleft |\vec{Q}|$  Typed copatterns  $\vec{Q}$  cover elimination of type A.

Result splitting:

$$\frac{A \triangleleft | \vec{Q} (\Delta \vdash q \Rightarrow B \rightarrow C)}{A \triangleleft | (\cdot \vdash \cdot \Rightarrow A)} \qquad \frac{A \triangleleft | \vec{Q} (\Delta \vdash q \Rightarrow B \rightarrow C)}{A \triangleleft | \vec{Q} (\Delta, x : B \vdash q x \Rightarrow C)}$$

$$\frac{A \triangleleft | \vec{Q} (\Delta \vdash q \Rightarrow \nu XR)}{A \triangleleft | \vec{Q} (\Delta \vdash q . d \Rightarrow R_d[\nu XR/X])_{d \in R}}$$

Variable splitting:

$$\frac{A \triangleleft | \vec{Q} (\Delta, x : A_1 \times A_2 \vdash q \Rightarrow C)}{A \triangleleft | \vec{Q} (\Delta, x_1 : A_1, x_2 : A_2 \vdash q[(x_1, x_2)/x] \Rightarrow C)}$$

$$\frac{A \triangleleft | \vec{Q} (\Delta, x : \mu XD \vdash q \Rightarrow C)}{A \triangleleft | \vec{Q} (\Delta, x' : D_c[\mu XD/X] \vdash q[c \ x'/x] \Rightarrow C)_{c \in D}}$$

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## Results

- Subject reduction.
- Progress: Any well-typed term that is not a value can be reduced.
- Thus, well-typed programs do not go wrong.



#### **Future Work**

- A productivity checker with sized types.
- Proof of strong normalization.



#### Conclusions

Accepted for presentation at POPL 2013:

Abel, Pientka, Thibodeau, and Setzer Copatterns – Programming Infinite Structures by Observation.

- Related Work:
  - Cockett et al. (1990s): Charity.
  - Zeilberger, Licata, Harper (2008): Focusing sequent calculus.