Normalization in Lambda-Calculus

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Untyped Lambda-Calculus

- Λ-terms and contexts:

$$\begin{array}{lcl} r, s, t, u, v & ::= & x \mid \lambda x t \mid t\, u \\ C & ::= & [] \mid \lambda x C \mid C\, u \mid t\, C \end{array}$$

- β-Contraction:

$$(\lambda x t)\, u \mapsto t[u/x]$$

- Full β-reduction: allow reduction in each subterm.

$$\frac{t \mapsto t'}{C[t] \longrightarrow C[t']}$$

- Multi-step reduction:

$$\begin{array}{ll} \longrightarrow^{+} & \text{transitive closure of } \longrightarrow \\ \longrightarrow^{*} & \text{reflexive-transitive closure of } \longrightarrow \end{array}$$

Normalization

Definition (Normal)

t is normal if it has no reduct, $t \not\longrightarrow$.

Definition (Weak normalization)

t is weakly normalizing (has a normal form) if $t \longrightarrow^* v \not\longrightarrow$.

Definition (Strong normalization, classically)

t is strongly normalizing if there exists no infinite reduction sequence $t \longrightarrow t_1 \longrightarrow t_2 \longrightarrow \dots$.

Definition (Strong normalization, constructively, inductively)

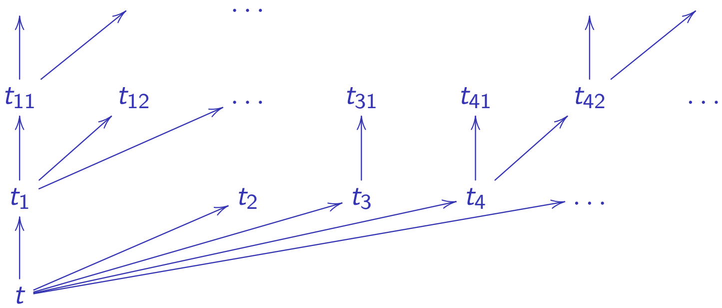
t is strongly normalizing if all of its reducts are strongly normalizing.

$$\frac{\{t' \mid t \longrightarrow t'\} \subseteq \mathsf{sn}}{t \in \mathsf{sn}}$$

Strong normalization, constructively

$$\frac{(t \longrightarrow \_) \subseteq \mathsf{sn}}{t \in \mathsf{sn}}$$

Intuitively: Each path in the reduction tree of t is finite.

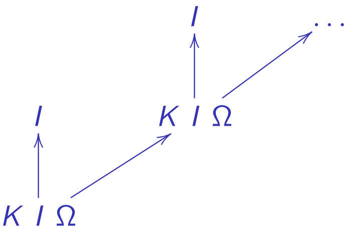


We say: the reduction tree is well-founded.

Leaves are normal forms.

Examples

- **Ex:** Any strongly normalizing Λ-term is weakly normalizing.
- Let $\Omega = (\lambda x.\, x\, x)\,(\lambda x.\, x\, x)$, $K = \lambda x \lambda y.\, x$, $I = \lambda x.\, x$.
- $\Omega \longrightarrow \Omega \longrightarrow \Omega \longrightarrow \dots$
- Ω admits an infinite reduction sequence ($\Omega \notin \mathsf{sn}$).
- $\Omega \longrightarrow t$ iff $t = \Omega$.
- Ω diverges/has no normal form.
- $K\ I\ \Omega$ is weakly, but not strongly normalizing.



Proving properties of sn

Theorem (Subterm)

Any subterm of a strongly normalizing term is strongly normalizing itself.

- (Does not hold for weak normalization, see $K\ I\ \Omega$.)
- Classical proof: Let $t = C[s] \in \mathsf{sn}$. Assume there is an infinite reduction sequence $s \longrightarrow s_1 \longrightarrow s_2 \longrightarrow \dots$. Then, there is also an infinite sequence $C[s] \longrightarrow C[s_1] \longrightarrow \dots$. Contradiction. So, $s \in \mathsf{sn}$.
- Constructive proof: Consider the reduction tree T of $t = C[s]$. We construct the reduction tree S of s by deleting all nodes (with subtrees) of T which are not of the form $C[s']$. Since T was well-founded, so is S.

Noetherian/wellfounded induction

Inductive definition of sn:

$$\frac{\forall t'.\ t \longrightarrow t' \implies t' \in \mathsf{sn}}{t \in \mathsf{sn}}$$

Definition (Noetherian/wellfounded induction)

To prove $\forall t \in \mathsf{sn}.\ P(t)$, we have the induction hypothesis

$$\forall t'.\ t \longrightarrow t' \implies P(t').$$

Meaning: to prove $P(t)$ we can use $P(t')$ for all reducts t' of t.

Intuition: if there are no infinite reduction sequences, we can view reducts as smaller.

Proving properties of sn by wellfounded induction

Theorem (Subterm)

Any subterm of a strongly normalizing term is strongly normalizing itself.
$C[s] \in \mathsf{sn} \implies s \in \mathsf{sn}$.

- Proof: By well-founded induction on $t \in \mathsf{sn}$, we show $P(t) :\equiv (\forall u.\, t = C[u] \implies u \in \mathsf{sn})$. Assume $t = C[s]$. To show $s \in sn$ it is sufficient to show $s' \in \mathsf{sn}$ for an arbitrary s' with $s \longrightarrow s'$. Since $t = C[s] \longrightarrow C[s']$ we have by induction hypothesis $P(C[s'])$. Choosing $u = s'$, with $C[s'] = C[s']$, we get $s' \in \mathsf{sn}$.

Inductive Characterization of Normal Forms

- Neutral (atomic) terms by rules:

$$\frac{}{x \Downarrow} \qquad \frac{r \Downarrow \qquad s \Uparrow}{r\, s \Downarrow}$$

- Normal terms by rules:

$$\frac{r \Downarrow}{r \Uparrow} \qquad \frac{t \Uparrow}{\lambda x t \Uparrow}$$

- Normal: $t \Uparrow$ iff $t \not\longrightarrow$.
- Neutral: $t \Downarrow$ iff $t \not\longrightarrow$ and t not a lambda-abstraction.

Closure Properties of sn

- If $t[s/x] \in \mathsf{sn}$, then $t \in \mathsf{sn}$.
- Neutral terms (implications, written as rules):

$$\frac{}{x \in \mathsf{sn}} \qquad \frac{r \in \mathsf{sn} \qquad r = x\,\vec{s} \qquad s \in \mathsf{sn}}{r\,s \in \mathsf{sn}}$$

- λs and weak head redexes:

$$\frac{t \in \mathsf{sn}}{\lambda x t \in \mathsf{sn}} \qquad \frac{s \in \mathsf{sn} \qquad s_1, \dots, s_n \in \mathsf{sn} \qquad t[s/x]\, s_1 \dots s_n \in \mathsf{sn}}{(\lambda x t)\, s\, s_1 \dots s_n \in \mathsf{sn}}$$

- Eliminating the ellipsis ...:

$$\boxed{\frac{u \in \mathsf{sn}}{(\lambda x t) u \longrightarrow_{\mathsf{sn}} t[u/x]} \qquad \frac{t \longrightarrow_{\mathsf{sn}} t'}{t\,u \longrightarrow_{\mathsf{sn}} t'\,u}} \qquad \frac{r \longrightarrow_{\mathsf{sn}} r' \qquad r' \in \mathsf{sn}}{r \in \mathsf{sn}}$$

Closure under Strong Head Expansion

Theorem

If $r \longrightarrow_{\mathsf{sn}} r'$ and $r' \in \mathsf{sn}$ then $r \in \mathsf{sn}$ and r not a λ.

Proof.

By induction on $r \longrightarrow_{\mathsf{sn}} r'$.

$$\frac{u \in \mathsf{sn}}{(\lambda x t) u \longrightarrow_{\mathsf{sn}} t[u/x]}$$

Have $t[u/x] \in \mathsf{sn}$. Side induction on (1) $t \in \mathsf{sn}$ and (2) $u \in \mathsf{sn}$. Show $(\lambda x t) u \longrightarrow s$ implies $s \in \mathsf{sn}$. Case $s = (\lambda x t') u$ covered by (1), $(\lambda x t) u'$ by (2), $t[u/x]$ by assumption.

$$\frac{t \longrightarrow_{\mathsf{sn}} t'}{t\, u \longrightarrow_{\mathsf{sn}} t'\, u}$$

By ind. hyp., $t \in \mathsf{sn}$ and t not a λ. Side induction on (1) $t \in \mathsf{sn}$ and (2) $u \in \mathsf{sn}$. Show $t\, u \longrightarrow s$ implies $s \in \mathsf{sn}$. Cases (1) $s = t''\, u$ and (2) $s = t\, u'$ covered accordingly. □

Inductive Characterization of Strongly Normalizing Terms

- Strongly normalizing neutral terms:

$$\frac{}{x \in \mathsf{SNe}} \qquad \frac{r \in \mathsf{SNe} \qquad s \in \mathsf{SN}}{r\, s \in \mathsf{SNe}}$$

- Strongly normalizing terms:

$$\frac{r \in \mathsf{SNe}}{r \in \mathsf{SN}} \qquad \frac{t \in \mathsf{SN}}{\lambda x t \in \mathsf{SN}} \qquad \frac{t \longrightarrow_{\mathsf{SN}} t' \qquad t' \in \mathsf{SN}}{t \in \mathsf{SN}}$$

- Strong head reduction:

$$\frac{u \in \mathsf{SN}}{(\lambda x t) u \longrightarrow_{\mathsf{SN}} t[u/x]} \qquad \frac{t \longrightarrow_{\mathsf{SN}} t'}{t\, u \longrightarrow_{\mathsf{SN}} t'\, u}$$

Soundness of SN

Theorem (Soundness of SN)

1. *If* $t \in \mathsf{SN}$ *then* $t \in \mathsf{sn}$.
2. *If* $t \in \mathsf{SNe}$ *then* $t \in \mathsf{sn}$ *and* $t = x\,\vec{s}$.
3. *If* $t \longrightarrow_{\mathsf{SN}} t'$ *then* $t \longrightarrow_{\mathsf{sn}} t'$.

Proof.

By induction on the derivation, using the closure properties of sn. □

Completeness of SN

Theorem (Completeness of SN)

1. *If* $t = x\,\vec{s} \in \mathsf{sn}$ *then* $x\,\vec{s} \in \mathsf{SNe}$.
2. *If* $t = (\lambda x r) s\,\vec{s} \in \mathsf{sn}$ *then* $t \longrightarrow_{\mathsf{SN}} r[s/x]\,\vec{s}$.
3. *If* $t \in \mathsf{sn}$ *then* $t \in \mathsf{SN}$.

Proof.

By lexicographic induction on the height of the reduction tree of t and the height of t. □

Simply-Typed Lambda-Calculus

- Type assignment to untyped terms:

$$\frac{(x\!:\!A) \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{\Gamma \vdash r : A \to B \qquad \Gamma \vdash s : A}{\Gamma \vdash r\,s : B} \qquad \frac{\Gamma, x\!:\!A \vdash t : B}{\Gamma \vdash \lambda x t : A \to B}$$

- Application difficult: $r, s \in \mathsf{SN} \not\Longrightarrow r\,s \in \mathsf{SN}$.
- Proof of strong normalization, outline:

$$\begin{array}{cl} \Gamma \vdash t : C & \\ \downarrow & \text{induction on types} \\ \Gamma \vdash t \Uparrow C & \\ \downarrow & \text{type erasure} \\ t \in \mathsf{SN} & \end{array}$$

Typed SN

- Typed version of SNe.

$$\frac{(x\!:\!A) \in \Gamma}{\Gamma \vdash x \Downarrow A} \qquad \frac{\Gamma \vdash r \Downarrow A \to B \qquad \Gamma \vdash s \Uparrow A}{\Gamma \vdash r\, s \Downarrow B}$$

- Typed version of SN.

$$\frac{\Gamma \vdash t \Downarrow C}{\Gamma \vdash t \Uparrow C} \quad \frac{\Gamma, x\!:\!A \vdash t \Uparrow B}{\Gamma \vdash \lambda x t \Uparrow A \to B} \quad \frac{\Gamma \vdash r \longrightarrow r' \Uparrow C \qquad \Gamma \vdash r' \Uparrow C}{\Gamma \vdash r \Uparrow C}$$

- Typed version of $\longrightarrow_{\mathsf{SN}}$.

$$\frac{\Gamma, x\!:\!A \vdash t : B \qquad \Gamma \vdash s \Uparrow A}{\Gamma \vdash (\lambda x t) s \longrightarrow t[s/x] \Uparrow B} \quad \frac{\Gamma \vdash r \longrightarrow r' \Uparrow A \to B \qquad \Gamma \vdash s : A}{\Gamma \vdash r\, s \longrightarrow r'\, s \Uparrow B}$$

Closure of typed SN under application

Theorem

If $\Gamma \vdash r \Uparrow A \to B$ *and* $\Gamma \vdash s \Uparrow A$ *then* $\Gamma \vdash r\, s \Uparrow B$.

- Interesting case:

$$\frac{\Gamma, x : A \vdash t \Uparrow B}{\Gamma \vdash \lambda x t \Uparrow A \to B} \qquad \Gamma \vdash s \Uparrow A$$

- To show $\Gamma \vdash (\lambda x t) s \Uparrow B$ we need $\Gamma \vdash t[s/x] \Uparrow B$.
- Follows from closure under substitution.
- Substitution could be tricky if $t = x\, u$: then $t[s/x] = s u$.
- Need again application thm., but type of u is smaller then type A of x.

Typed SN is closed under substitution

Lemma (Substitution)

Let $\Gamma \vdash s \Uparrow A$.

1. *If* $\Gamma, x{:}A, \Gamma' \vdash r \Downarrow C$ *then either* $\Gamma, \Gamma' \vdash r[s/x] \Downarrow C$ *or* $\Gamma, \Gamma' \vdash r[s/x] \Uparrow C$ *and* C *is smaller than* A.
2. *If* $\Gamma, x{:}A, \Gamma' \vdash r \Uparrow C$ *then* $\Gamma, \Gamma' \vdash r[s/x] \Uparrow C$.
3. *If* $\Gamma, x{:}A, \Gamma' \vdash r \longrightarrow r' \Uparrow C$ *then* $\Gamma, \Gamma' \vdash r[s/x] \longrightarrow r'[s/x] \Uparrow C$.

Proof.

Simultaneously by main induction on A and side induction on the derivation. □

Strong normalization for simple types

Theorem

If $\Gamma \vdash t : C$ *then* $t \in \mathsf{SN}$.

Proof.

Prove $\Gamma \vdash t \Uparrow C$ by induction on the type derivation, using closure under application. Then, erase to $t \in \mathsf{SN}$. □

Further details and Twelf formalization: [Abel, LFM 2004].
Related: hereditary substitutions [Watkins et al., 2002].

Intersection Types

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash t : B}{\Gamma \vdash t : A \cap B} \qquad \frac{\Gamma \vdash r : A \cap B}{\Gamma \vdash r : A} \qquad \frac{\Gamma \vdash r : A \cap B}{\Gamma \vdash r : B}$$

- STL with intersection types is strongly normalizing.
- Any strongly normalizing term can be typed with intersections.

$$t \in \mathsf{SN} \iff \exists \Gamma, A.\ \Gamma \vdash t : A$$

- Example: $\lambda x.\, x\, x : (A \cap (A \to B)) \to B$.

SN for intersection types

- Add rules for $\cap$-elimination to $\Downarrow$:

$$\frac{\Gamma \vdash r \Downarrow A \cap B}{\Gamma \vdash r \Downarrow A} \qquad \frac{\Gamma \vdash r \Downarrow A \cap B}{\Gamma \vdash r \Downarrow B}$$

- Add rules for $\cap$-introduction to $\Uparrow$:

$$\frac{\Gamma \vdash t \Uparrow A \qquad \Gamma \vdash t \Uparrow B}{\Gamma \vdash t \Uparrow A \cap B}$$

Lemma (Closure under $\cap$-elimination)

If $\Gamma \vdash t \Uparrow A \cap B$ then $\Gamma \vdash t \Uparrow A$ and $\Gamma \vdash t \Uparrow B$ [Abel, HOR 2007].

Completeness of Intersection Types for SN

Lemma (Anti-substitution)

Let $\Gamma \vdash s : A_0$. *If* $\Gamma \vdash t[s/x] : C$ *then* $\Gamma, x{:}A \vdash t : C$ *and* $\Gamma \vdash s : A$ *for some* A.

For instance $y : \mathbb{N} \to A \vdash y\,0 : A$ and $y : B \vdash y[y\,0/x] : B$. Have $y : B \cap (\mathbb{N} \to A) \vdash y\,0 : A$ and $y : B \cap (\mathbb{N} \to A), x : A \vdash y : B$. Thus, $y : B \cap (\mathbb{N} \to A) \vdash (\lambda xy)(y\,0) : B$ (subject expansion).

Theorem

1. *If* $r \in \mathsf{SNe}$ *then* $\Gamma \vdash r : X$ *for some* Γ *and type variable* X.
2. *If* $t \in \mathsf{SN}$ *then* $\Gamma \vdash t : A$ *for some* Γ, A.
3. *If* $t \longrightarrow_{\mathsf{SN}} t'$ *and* $\Gamma' \vdash t' : C$ *then* $\Gamma \vdash t : C$ *for some* Γ.