Wellfounded Recursion with Copatterns

Andreas Abel

Department of Computer Science and Engineering
Chalmers and Gothenburg University, Sweden

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This is joint work with Brigitte Pientka.
Short CV

1994– Studying Computer Science at Ludwig-Maximilians-University Munich (LMU)
1999 Diploma (supervisor: Thorsten Altenkirch)
1999– Doctoral studies at LMU
2000-01 Research stay at Carnegie-Mellon-University, Pittsburgh (advisor: Frank Pfenning)
2004-05 Postdoc at Chalmers, Gothenburg (Coquand, Dybjer, Hughes, Sheeran)
2005-13 Assistent to Martin Hofmann at LMU
2006 PhD (Dr. rer. nat.) from LMU (supervisor: Martin Hofmann)
2009-10 Research stay at INRIA/PPS, Paris, France (Curien, Herbelin)
2013 Habilitation at LMU
2013– Senior Lecturer at Gothenburg University
Research Interests

- Programming Languages and Type Systems
- Semantics
- Verification
- (Type-Based) Termination
- Dependent Types
- Implementing Agda
Productivity Checking

- **Coinductive** structures: streams, processes, servers, continuous computation.
- Productivity: each request returns an answer after some time.
- Request on stream: *give me the next element.*
- Dependently typed languages have a **productivity checker**:

\[
nats = 0 :: \text{map (1 + \_)} \ nats
\]

- Rejected by Coq and Agda’s syntactic guardedness check.
Fibonacci Stream

- Recurrence for Fibonacci numbers:

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</thead>
<tbody>
<tr>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
</tr>
<tr>
<td>adds 0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
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<td>1</td>
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<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

- Elegant implementation:

\[
\text{fib} = 0 :: 1 :: \text{adds fib (fib.tail)}
\]

- Rejected by guardedness check.
Coinduction and Dependent Types

- Consider the corecursively defined stream $a :: a :: a :: \ldots$

  $$\text{repeat } a = a :: \text{repeat } a$$

- A dilemma:
  - Checking dependent types needs strong reduction.
  - Corecursion needs lazy evaluation.

- The current compromise (Coq, Agda):
  Corecursive definitions are unfolded only under elimination.

  $$\text{repeat } a \nrightarrow (\text{repeat } a).\text{tail} \rightarrow (a :: \text{repeat } a).\text{tail} \rightarrow \text{repeat } a$$

- Reduction is context-sensitive.
Issues with Context-Sensitive Reduction

- Subject reduction is lost (Giménez 1996, Oury 2008).
- The Fibonacci stream is still diverging:

\[ \text{fib} = 0 :: 1 :: \text{adds fib (fib.tail)} \]

\[ \text{fib.tail} \rightarrow 1 :: \text{adds fib (fib.tail)} \]
\[ \rightarrow 1 :: \text{adds fib (1 :: \text{adds fib (fib.tail)})} \]
\[ \rightarrow \ldots \]

- At POPL 2013, we presented a solution:

  A. Abel, B. Pientka, D. Thibodeau, and A. Setzer.  
  **Copatterns**: Programming infinite structures by observations.  
Copatterns — The Principle

- Define infinite objects (streams, functions) by observations.
- A function is defined by its applications.
- A stream by its head and tail.

\[
\text{repeat } a \ . \text{head} = a \\
\text{repeat } a \ . \text{tail} = \text{repeat } a
\]

- These equations are taken as reduction rules.
- \text{repeat } a \ does \ not \ reduce \ by \ itself.
- No extra laziness required.
Deep Observations

- Any covering set of observations allowed for definition:

  \[
  \begin{align*}
  \text{fib.head} & = 0 \\
  \text{fib.tail.head} & = 1 \\
  \text{fib.tail.tail} & = \text{adds fib (fib.tail)}
  \end{align*}
  \]

- Now \text{fib.tail} is stuck. Good!

<table>
<thead>
<tr>
<th>Depth</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observations</td>
<td>id</td>
<td>.head</td>
<td>.tail.head</td>
<td>.tail.tail</td>
</tr>
</tbody>
</table>

Abel (Chalmers)
Stream Productivity

Definition (Productive Stream)
A stream is **productive** if all observations on it converge.

- Example of non-productiveness:
  
  \[
  \text{bla} = 0 :: \text{bla} . \text{tail}
  \]

- Observation \text{bla} . \text{tail} diverges.

- This is apparent in copattern style...
  
  \[
  \begin{align*}
  \text{bla} . \text{head} & = 0 \\
  \text{bla} . \text{tail} & = \text{bla} . \text{tail}
  \end{align*}
  \]
Theorem (repeat is productive)

repeat $a \cdot \text{tail}^n$ converges for all $n \geq 0$.

Proof.

By induction on $n$.

Base (repeat $a$).tail$^0 = \text{repeat } a$ does not reduce.

Step (repeat $a$).tail$^{n+1} = (\text{repeat } a).\text{tail}.\text{tail}^n \rightarrow (\text{repeat } a).\text{tail}^n$ which converges by induction hypothesis.
Productive Functions

Definition (Productive Function)
A function on streams is productive if it maps productive streams to productive streams.

\[(\text{adds } s \ t).\text{head} = s.\text{head} + t.\text{head}\]
\[(\text{adds } s \ t).\text{tail} = \text{adds} (s.\text{tail}) (t.\text{tail})\]

- **Productivity** of `adds` not sufficient for `fib`!
- Malicious `adds`:

\[
\begin{align*}
\text{adds'} s \ t & = t.\text{tail} \\
\text{fib.tail.tail} & \rightarrow \text{adds'} \text{ fib (fib.tail)} \\
& \rightarrow \text{fib.tail.tail} \rightarrow \ldots
\end{align*}
\]
**i-Productivity**

**Definition (Productive Stream)**
A stream $s$ is $i$-productive if all observations of depth $< i$ converge. Notation: $s : \text{Stream}^i$.

**Lemma**
adds : $\text{Stream}^i \rightarrow \text{Stream}^i \rightarrow \text{Stream}^i$ *for all* $i$.

**Theorem**
$\text{fib}$ *is* $i$-productive *for all* $i$.

**Proof, case $i + 2$:** Show $\text{fib}$ is $(i + 2)$-productive.

Show $\text{fib.} \text{tail.} \text{tail}$ *is* $i$-productive.

IH: $\text{fib}$ is $(i + 1)$-productive, so $\text{fib}$ *is* $i$-productive. (Subtyping!)

IH: $\text{fib}$ is $(i + 1)$-productive, so $\text{fib.} \text{tail}$ *is* $i$-productive.

By Lemma, adds $\text{fib} (\text{fib.} \text{tail})$ *is* $i$-productive.
Type System for Productivity

- “Church $\mathcal{F}^\omega$ with inflationary and deflationary fixed-point types”.
- Coinductive types = deflationary iteration:

$$\text{Stream}^i A = \bigcap_{j<i} (A \times \text{Stream}^j A)$$

- Bidirectional type-checking:
- Type inference $\Gamma \vdash r \Rightarrow A$ and checking $\Gamma \vdash t \Leftarrow A$.

$$\Gamma \vdash r \Rightarrow \text{Stream}^i A$$

$$\Gamma \vdash r . \text{tail} \Rightarrow \forall j<i. \text{Stream}^j A \quad \Gamma \vdash a < i$$

$$\Gamma \vdash r . \text{tail} a : \text{Stream}^a A$$
Copattern typing

Fibonacci again (official syntax with explicit sizes).

\[
fib : \forall i. |i| \Rightarrow \text{Stream}^i \mathbb{N}
\]

\[
fib i . \text{head} j = 0
\]

\[
fib i . \text{tail} j . \text{head} k = 1
\]

\[
fib i . \text{tail} j . \text{tail} k = \text{adds} k (\text{fib} k) (\text{fib} j . \text{tail} k)
\]

Copattern inference \(\Delta | A \vdash \vec{q} \Rightarrow C\) (linear).

\[
\begin{array}{c}
\cdot \mid \text{Stream}^k \mathbb{N} \vdash \quad \cdot \Rightarrow \text{Stream}^k \mathbb{N} \\
\hline
k < j \mid \forall k < j. \text{Stream}^k \mathbb{N} \vdash \quad k \Rightarrow \text{Stream}^k \mathbb{N} \\
\hline
k < j \mid \text{Stream}^j \mathbb{N} \vdash \quad \text{.tail} k \Rightarrow \text{Stream}^k \mathbb{N} \\
\hline
j < i, k < j \mid \forall j < i. \text{Stream}^j \mathbb{N} \vdash \quad j . \text{tail} k \Rightarrow \text{Stream}^k \mathbb{N} \\
\hline
j < i, k < j \mid \text{Stream}^i \mathbb{N} \vdash \quad \text{.tail} j . \text{tail} k \Rightarrow \text{Stream}^k \mathbb{N}
\end{array}
\]

Type of recursive call \(fib : \forall i' < i. \text{Stream}^{i'} \mathbb{N}\)
Conclusions

- A unified approach to termination and productivity: Induction.
  - Recursion as induction on data size.
  - Corecursion as induction on observation depth.
- Adaptation of sized types to deep (co)patters:
  - Shift to in-/deflationary fixed-point types.
  - Bounded size quantification.
- Implementations:
  - MiniAgda: ready to play with!
  - Agda (with James Chapman): in development version, planned for next release (2.3.4).

Andreas Abel and Brigitte Pientka.
Wellfounded recursion with copatterns:
A unified approach to termination and productivity.

*International Conference on Functional Programming (ICFP 2013).*
Some Related Work

- Sized types: many authors (1996–)
- Inflationary fixed-points: Dam & Sprenger (2003)
- Observation-centric coinduction and coalgebras: Hagino (1987), Cockett & Fukushima (Charity, 1992)
- Form of termination measures taken from Xi (2002)
Pattern typing rules

\[ \Delta; \Gamma \vdash_{\Delta_0} p \mathbin{\triangleq} A \]

Pattern typing (linear).

In: \( \Delta_0, p, A \) with \( \Delta_0 \vdash A \). Out: \( \Delta, \Gamma \) with \( \Delta_0, \Delta; \Gamma \vdash p \mathbin{\triangleq} A \).

\[ \cdot; x: A \vdash_{\Delta_0} x \mathbin{\triangleq} A \]
\[ \cdot; \cdot \vdash_{\Delta_0} () \mathbin{\triangleq} 1 \]

\[ \Delta_1; \Gamma_1 \vdash_{\Delta_0} p_1 \mathbin{\triangleq} A_1 \]
\[ \Delta_2; \Gamma_2 \vdash_{\Delta_0} p_2 \mathbin{\triangleq} A_2 \]

\[ \Delta_1, \Delta_2; \Gamma_1, \Gamma_2 \vdash_{\Delta_0} (p_1, p_2) \mathbin{\triangleq} A_1 \times A_2 \]

\[ \Delta; \Gamma \vdash_{\Delta_0} p \mathbin{\triangleq} \exists j < a^\uparrow. S_c (\mu^j S) \]

\[ \Delta; \Gamma \vdash_{\Delta_0} c p \mathbin{\triangleq} \mu^a S \]

\[ \Delta; \Gamma \vdash_{\Delta_0} X: \kappa \vdash \exists \kappa F \]

\[ X: \kappa, \Delta; \Gamma \vdash_{\Delta_0} X p \mathbin{\triangleq} \exists \kappa F \]
Copattern typing rules

Pattern spine typing. In: $\Delta_0, A, \bar{q}$ with $\Delta_0 \vdash A$.
Out: $\Delta, \Gamma, C$ with $\Delta_0, \Delta; \Gamma \vdash C$ and $\Delta_0, \Delta; \Gamma, z: A \vdash z \bar{q} \Rightarrow C$.

\[ \frac{}{\Delta; \Gamma \mid A \vdash_{\Delta_0} \bar{q} \Rightarrow C} \]

\[ \frac{}{\Delta_1; \Gamma_1 \vdash_{\Delta_0} p \Leftarrow A \quad \Delta_2; \Gamma_2 \mid B \vdash_{\Delta_0} \bar{q} \Rightarrow C} \]
\[ \Delta_1, \Delta_2; \Gamma_1, \Gamma_2 \mid A \rightarrow B \vdash_{\Delta_0} p \bar{q} \Rightarrow C \]

\[ \frac{}{\Delta; \Gamma \mid \forall j < a^\uparrow. \ R_d (\nu^j R) \vdash_{\Delta_0} \bar{q} \Rightarrow C} \]

\[ \frac{}{\Delta; \Gamma \mid \nu^a R \vdash_{\Delta_0} d \bar{q} \Rightarrow C} \]

\[ \frac{}{\Delta; \Gamma \mid F @^\kappa X \vdash_{\Delta_0, X: \kappa} \bar{q} \Rightarrow C} \]
\[ X: \kappa, \Delta; \Gamma \mid \forall_\kappa F \vdash_{\Delta_0} X \bar{q} \Rightarrow C \]
Semantics

- Reduction:

\[
\begin{align*}
\frac{\vec{e} / \vec{q} \sigma}{\lambda\{\vec{q} \rightarrow t\} \vec{e} \vec{e}' \mapsto t\sigma \vec{e}'} \\
\frac{\lambda D_k \vec{e} \mapsto t}{f \vec{e} \mapsto t} (f : A = \vec{D}) \in \Sigma
\end{align*}
\]

- Types are reducibility candidates \( A \):
  - \( A \) is a set of strongly normalizing terms.
  - \( A \) is closed under reduction.
  - \( A \) is closed under addition of well-behaved neutrals (redexes and terminally stuck terms).
  - \( A \) is closed under simulation:
    - \( r \) is simulated by \( r_{1..n} \) if \( r \vec{e} \mapsto t \) implies \( r_k \vec{e} \mapsto t \) for some \( k \).