Computational Aspects of Covering in Dominance Graphs

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22nd Conference on Artificial Intelligence

Outline

Preliminaries: Dominance Graphs and Choice Sets

Choice Sets Based on Covering

Computing The Choice Sets

Some Set-Theoretic Relationships

Dominance Graphs and Choice Sets

- Various problems in AI and MASs can be cast as finding "most desirable" alternatives according to a binary relation
  - Valid arguments
  - Socially preferred candidates
  - Winners of a competition
  - Optimal strategies in a symmetric two-player zero-sum game
  - Feasible coalitions
- Can be viewed as a (directed) *dominance graph*
- Maximality not well-defined in the presence of cycles (termed *Condorcet cycles* in social choice)
- Various *solution concepts* (or choice sets) take over the role of maximality
- This talk: choice sets based on *covering*

Some Notation

- Finite set A of *alternatives*
- Asymmetric and irreflexive *dominance relation* $\succ \subseteq A \times A$
- $a \succ b$ means that a "is strictly better than" b or "beats" b in a pairwise comparison
- We do not assume completeness or transitivity of $\succ$ but allow for ties and cycles
- Tournament: a complete dominance relation
- Choice set: a function $f : (A, \succ) \to 2^A$

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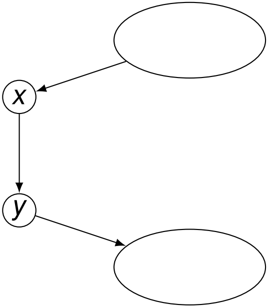
- Alternative view: *adjacency game* $\Gamma(A, \succ) = (\{0, 1\}, A, p)$ where

$$p(a, b) = \begin{cases} (1, -1) & \text{if } a \succ b \\ (-1, 1) & \text{if } b \succ a \\ (0, 0) & \text{otherwise} \end{cases}$$

Covering Relations



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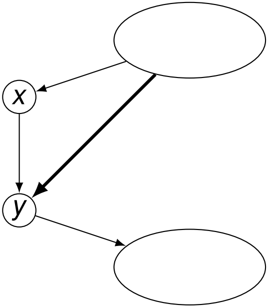

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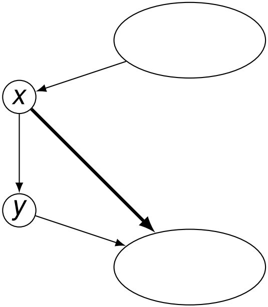

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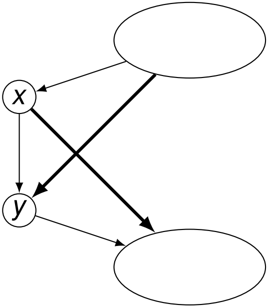

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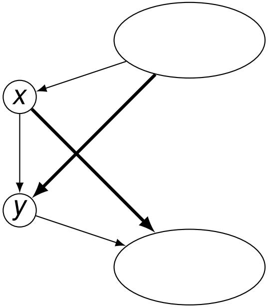

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- Tournaments: All three notions of covering coincide

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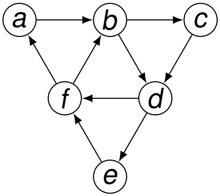

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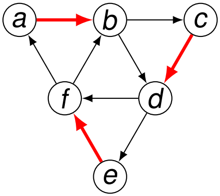

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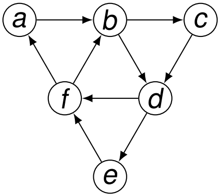

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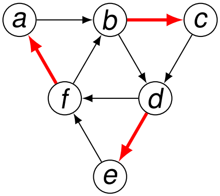

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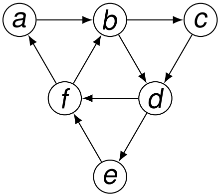

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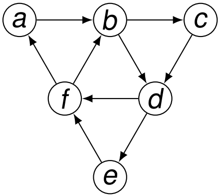

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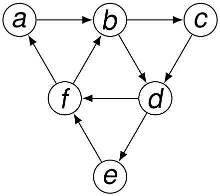

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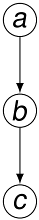

- Computation very easy and parallelizable
- Not idempotent, can be iterated to obtain smaller choice sets

Covering Sets

- Again consider a covering relation C
- $B \subseteq A$ is a *covering set* under C if
  (i) $UC_C(B) = B$, and
  (ii) for all $x \in A \setminus B$, $x \notin UC_C(B \cup \{x\})$
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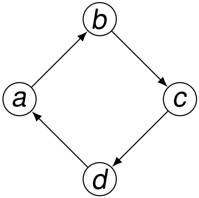
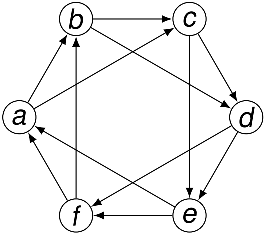
- Properties (i) and (ii) are called internal and external stability
- *Minimal covering set (MC)*: a covering set that is minimal w.r.t. set inclusion
- There exists a unique bidirectional MC (Dutta, 1988; Dutta & Laslier, 1999; Peris & Subiza, 1999)
- Axiomatization: smallest Condorcet choice set satisfying SSP, γ^*, and CDP (Peris & Subiza, 1999)
- Positive foundation (in tournaments): coincides with Shapley's weak saddle of the adjacency game (Duggan & LeBreton, 1996)

Minimal Covering Sets





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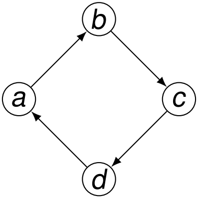

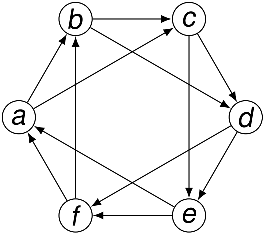

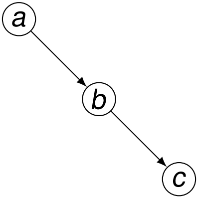

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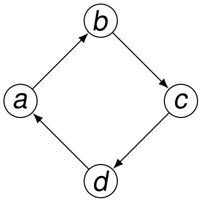

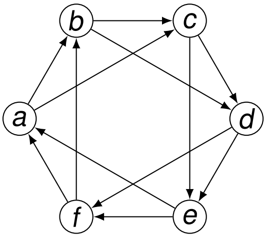

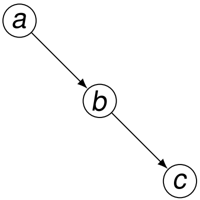

- Minimal upward and downward covering sets need not be unique
- Downward covering set may not exist
- **Theorem:** There always exists a minimal upward covering set
- Proof idea: show (by induction) that $UC_u^k(A)$ is externally stable for every k

Computing Minimal Covering Sets

- **Theorem:** *MC* can be computed in polynomial time.

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  - **procedure** $MC(A, \succ)$
    $B \leftarrow ES(A)$
    loop
    $A' \leftarrow \{\, a \in A \setminus B \mid a \text{ uncovered in } B \cup \{a\} \,\}$
    if $A' = \emptyset$ **then return** B **end if**
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    end loop
  - Show that $B \subseteq MC(A)$ at any time (by induction on $|B|$)
  - For this, show that every element of $MC(A')$ has to be part of every superset of B that is covering for A
  - The rest is a case analysis

The Missing Link

- Essential set $ES(A)$: set of alternatives in the support of *some* Nash equilibrium of $\Gamma(A, \succ)$
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- Proof sketch:
  - Show that $ES(A)$ coincides with support of the unique *quasi-strict equilibrium* of $\Gamma(A, \succ)$
  - Construct a linear program for finding a quasi-strict equilibrium in symmetric zero-sum games
  - LP can be solved in polynomial time (Khachiyan, 1979)

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procedure $ES(A, \succ)$

maximize ε

subject to

$$\begin{aligned} &\textstyle\sum_{j \in A} s_j \cdot m_{ij} \leq 0 && \forall i \in A \\ &\textstyle\sum_{j \in A} s_j = 1 && \\ &s_j \geq 0 && \forall j \in A \\ &s_i - \textstyle\sum_{j \in A} s_j \cdot m_{ij} - \varepsilon \geq 0 && \forall i \in A \end{aligned}$$

return $\{ a \in A \mid s_a > 0 \}$

Unidirectional Covering

- Minimal upward or downward covering sets can be more discriminating than *MC*
- **Theorem:** Deciding whether
  - an alternative is contained in some minimal upward covering set
  - an alternative is contained in some minimal downward covering set
  - there exists a downward covering set

  is NP-hard
- Proof idea: reductions from SAT
- We have some mild evidence that the first two problems are actually Θ_2^P-complete (like Kemeny, Dodgson, and Young)

Relationships

- For every C, $MC_C(A) \subseteq UC_C^\infty(A)$
- $UC_u(A)$ and $UC_d(A)$ can have an empty intersection
- $MC(A)$ is upward and downward covering
- There may be additional upward or downward covering sets not intersecting with MC

Relationships With Even More Choice Sets

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- $V \subseteq A$ is a *(von Neumann-Morgenstern) stable set* if
  - (i) $a \succ b$ for no $a, b \in V$ and
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- **Theorem:** Every stable set is a minimal upward covering set
- $a \in A$ is in the *Banks set* of A if there exists $X \subseteq A$ such that $\succ$ is complete and transitive on X with maximal element a and there is no $b \in A$ such that $b \succ x$ for all $x \in X$
- **Theorem:** The Banks set intersects with every downward covering set

Conclusion

- Finding desirable elements according to a binary relation is an important problem in AI and MASs
- Choice sets take over the role of maximal elements if the relation is not transitive
- Choice sets based on covering relations: uncovered set, minimal covering set
- The minimal (bidirectional) covering set has nice properties and can be computed efficiently
- Minimal upward or downward covering sets may not be unique and deciding membership is NP-hard
- Upward and downward covering sets are related to stable sets and the Banks set, respectively

Thank you for your attention!