On the Complexity of Finding Pure Nash Equilibria in Strategic Games

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Strategic Games

- Game $\Gamma = (N, (C_i)_{i \in N}, (u_i)_{i \in N})$
  - N a set of *players*
  - C_i a nonempty set of *(pure) strategies* for player i
  - $u_i : C \to \mathbb{R}$ a *payoff function* for player i and *(pure) strategy profiles* $C = \times_{j \in N} C_j$
- Examples: prisoners' dilemma, matching pennies

	C	D
C	-1,-1	-5,0
D	0,-5	-4,-4

	H	T
H	1,0	0,1
T	0,1	1,0

Nash Equilibrium

- *Mixed strategy profile* $\sigma \in \times_{i \in N} \Delta(C_i)$, where $\Delta(C_i)$ is a probability distribution over C_i
- Strategy profile (σ_{-i}, τ_i) where the ith component is $\tau_i \in \Delta(C_i)$ and all other components are as in σ
- σ is a *Nash equilibrium* if the following holds for every player $i \in N$ and every $\tau_i \in \Delta(C_i)$:

$$u_i(\sigma) \geq u_i(\sigma_{-i}, \tau_i)$$

- Equivalently:

$$\text{if } \sigma_i(c_i) > 0, \text{ then } c_i \in \arg\max_{d_i \in C_i} u_i(\sigma_{-i}, [d_i]),$$

  where $[d_i] \in \Delta(C_i)$ puts probability 1 on d_i
- General existence theorem (Nash 1951): Any finite game Γ has at least one equilibrium in $\times_{i \in N} \Delta(C_i)$

Complexity of Finding Nash Equilibria

- Mixed strategies: PPAD complete for $|N| \geq 2$ (Chen and Deng, 2005)
- Pure Nash equilibria can be found by enumeration of pure strategy profiles
- Number of pure strategy profiles is polynomial in $|C_i|$, exponential in $|N|$
- Succinct representation required to show high complexity

Games in Graphical Normal Form

- Payoff of a player depends only on strategies played by a *subset* of the other players
- Game $\Gamma = (N, (C_i)_{i \in N}, (neigh_i)_{i \in N}, (u_i)_{i \in N})$
  - N a set of *players*
  - C_i a nonempty set of *(pure) strategies* for i
  - $neigh_i \subseteq N \setminus i$ the *neighbourhood* of i
  - $u_i : C_i \times (\times_{j \in neigh_i} C_j) \to \mathbb{R}$ a *payoff function* for i
- Γ succinctly representable if for all i, $|neigh_i|$ is bounded by a constant

Complexity Results about Pure Nash Equilibria

Theorem

Deciding whether a strategic game Γ *has a pure strategy Nash equilibrium is NP-complete. Hardness holds even if* Γ *is in GNF, and* $|C_i| \leq 2$, $|neigh_i| \leq 2$, $|\{u_i(c) | c \in C\}| \leq 2$ *for all* i.

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Theorem

Deciding whether a strategic game Γ in GNF with $|neigh_i| \leq 1$ for all i has a pure strategy Nash equilibrium is NL-complete. Hardness holds even if $|\{u_i(c)|c \in C\}| \leq 2$ for all i.

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Theorem

Deciding whether a strategic game Γ *in GNF with* $|neigh_i| \leq 1$ *for all* i *has a pure strategy Nash equilibrium is NL-complete. Hardness holds even if* $|\{u_i(c) | c \in C\}| \leq 2$ *for all* i.

Theorem

Deciding whether a strategic game Γ *in GNF with* $|neigh_i| \leq 1$ *and* $|C_i| \leq k$ *for all* i *and some constant* k *has a pure strategy Nash equilibrium is L-complete. Hardness holds even if* $|\{u_i(c) | c \in C\}| \leq 2$ *for all* i.

Thank you for your attention!