Voting Caterpillars

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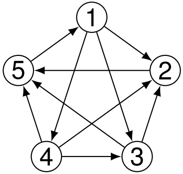
Choosing from a Tournament

- Set $A = \{1, 2, \ldots, m\}$ of *alternatives*
- *Tournament* $T \in \mathcal{T}(A)$: a complete, irreflexive, asymmetric relation on A
- Directed edge (a, b) means that a "beats" b



- For example arises from majority voting over pairs of alternatives (with an odd number of voters, linear preferences)

- *Tournament solution* $f : \mathcal{T}(A) \to 2^A$ that singles out good alternatives in the presence of cycles

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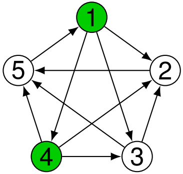

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- *Tournament solution* $f : \mathcal{T}(A) \to 2^A$ that singles out good alternatives in the presence of cycles
- *Copeland* solution: alternatives with maximum (out-)degree

Voting Trees

- A procedure for choosing from a tournament
- *Voting tree* Γ on A: Binary tree with elements of A at the leaves
- Given tournament T, label each internal node with the label of its children that is better according to T
- Label at the root is the winner, denoted $\Gamma(T)$
- Question: Which solutions can be implemented by voting trees?
- Γ *implements* f if for any $T \in \mathcal{T}(A)$, $\Gamma(T) \in f(T)$

- Copeland solution can be implemented if and only if $m \leq 7$ (Moulin, 1986; Srivastava and Trick, 1996)
- Question: Can the Copeland solution be *approximated*?

Two Models

- *Deterministic*: Voting tree Γ on A provides approximation ratio α if for all $T \in \mathcal{T}(A)$,

$$\frac{s_{\Gamma(T)}}{\max_{i \in A} s_i(T)} \geq \alpha,$$

where s_i is the degree (or *score*) of i

- *Randomized*: Probability distribution Δ over voting trees on A
  - provides approximation ratio α if for all $T \in \mathcal{T}(A)$,

$$\frac{\mathbb{E}_{\Gamma \sim \Delta}[s_{\Gamma(T)}]}{\max_{i \in A} s_i(T)} \geq \alpha$$

  - is *admissible* if its support contains only *surjective* trees

Upper Bounds by Composition Consistency

- **Theorem:** No voting tree provides an approximation ratio better than $\frac{3}{4} + O(\frac{1}{m})$.
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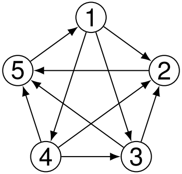
- $C \subseteq A$ is a *component* of $T \in \mathcal{T}(A)$ if for all $i, j \in C$, $k \in A \setminus C$, iTk if and only if jTk



- **Lemma (Moulin, 1986):** Consider $T, T' \in \mathcal{T}(A)$ that differ only inside a component C. Then for any voting tree Γ on A,
  - (i) $\Gamma(T) \in C$ if and only if $\Gamma(T') \in C$
  - (ii) $\Gamma(T) \in A \setminus C$ implies $\Gamma(T) = \Gamma(T')$

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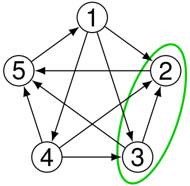

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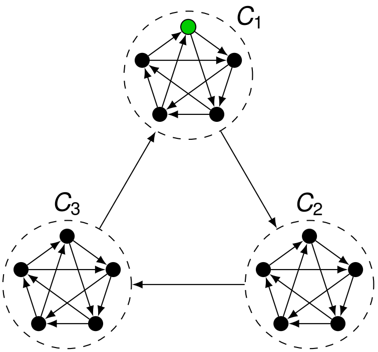
Proof Sketch

- Choose $m = 3k$ for k odd
- T: three-cycle of regular components of size k
- $s_{\Gamma(T)} = k + \frac{k-1}{2}$
- W.l.o.g., $\Gamma(T) \in C_1$
- Now define T' by making C_2 transitive



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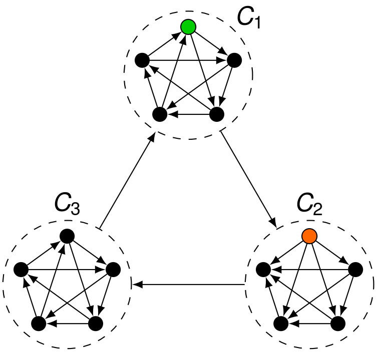

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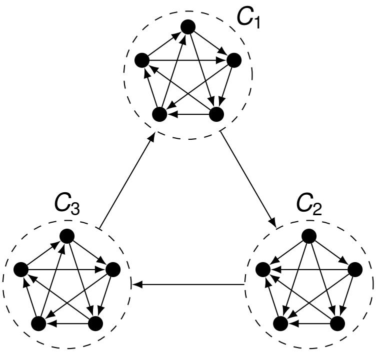


- Randomized upper bound: use Yao's principle

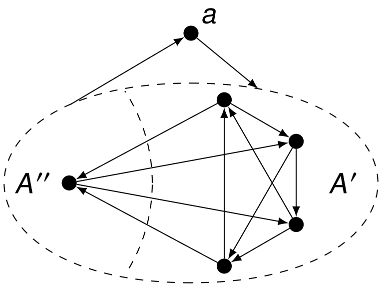
A Randomized Lower Bound

- **Theorem:** There exists an admissible randomization over voting trees of polynomial size with an approximation ratio of $\frac{1}{2} - O(\frac{1}{m})$.
- Trivial for non-admissible randomizations, random alternative has expected degree $\frac{m-1}{2}$
- Proof uses voting caterpillars
- 1-caterpillar: a leaf
- k-caterpillar: a binary tree, children of the root are a $(k-1)$-caterpillar and a leaf
- k-RC: leafs chosen uniformly i.i.d.

Proof Outline

- k-RC is close to an admissible distribution
- Equivalent to a random walk on the tournament
  - move from i to *better* alternative j with probability $p_{ij} = \frac{1}{m}$
  - stay put with probability $p_{ii} = \frac{s_i+1}{m}$
- Stationary distribution π such that $\pi_i = \sum_j \pi_j p_{ji}$
- Yields expected degree $\sum_{i \in A} \pi_i s_i \geq \frac{m-1}{2}$
- Fast convergence:
  - Look at reversibilization M of the transition matrix
  - Fill (1991): $4\|\pi^{(k)} - \pi\|^2 \leq m(\beta_1(M))^k$, where $\beta_1(M)$ is the second largest eigenvalue of M
  - Sinclair and Jerrum (1989): $1 - 2\Phi \leq \beta_1(A) \leq 1 - \frac{\Phi^2}{2}$, where Φ is the *conductance* of M

The Analysis is Tight



- $A' \cup A''$ regular
- $|A''| = \epsilon(m-1)$
- $\pi_a = \frac{\sum_{j:aTj} \pi_j}{m-s_a-1} \leq \frac{1}{m-s_a-1} \leq \frac{1}{\epsilon(m-1)}$
- $\sum_i \pi_i s_i \leq \frac{1}{\epsilon(m-1)}(m-1) + \frac{\epsilon(m-1)-1}{\epsilon(m-1)} \cdot \left(\frac{m-1}{2} + 1\right) \leq \frac{m-1}{2} + \frac{1}{\epsilon} + 1$
- This counterexample is generic, so we either get $\frac{1}{2}$ w.h.p. or something better in expectation

What We (Don't) Know

- Permutation tree: balanced tree, every alternative at one leaf
- Trivial (deterministic) lower bound of $\Theta(\frac{\log m}{m})$
- Large gap between this and the upper bound of $\frac{3}{4}$
- Balanced trees of height $(\log m) + 1$ do not help
- Composition of permutation trees cannot do better than $\frac{1}{2}$

- Randomized model: gap between $\frac{1}{2}$ and $\frac{5}{6}$
- Randomized balanced trees "oscillate", don't provide any bound
- Higher-order caterpillars also oscillate

Thank you!