Ehrenfeucht-Fraïssé Games on Omega-Terms

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Abstract

Fragments of first-order logic over words can often be characterized in terms of finite monoids or finite semigroups. Usually these algebraic descriptions yield decidability of the question whether a given regular language is definable in a particular fragment. An effective algebraic characterization can be obtained from identities of so-called omega-terms. In order to show that a given fragment satisfies some identity of omega-terms, one can use Ehrenfeucht-Fraïssé games on word instances of the omega-terms. The resulting proofs often require a significant amount of bookkeeping with respect to the constants involved. In this paper we introduce Ehrenfeucht-Fraïssé games on omega-terms. To this end we assign a labeled linear order to every omega-term. Our main theorem shows that a given fragment satisfies some identity of omega-terms if and only if Duplicator has a winning strategy for the game on the resulting linear orders.

As an application of our main result, we show that one can decide in exponential time whether all aperiodic monoids satisfy some given identity of omega-terms, thereby improving a result of McCammond (Int. J. Algebra Comput., 2001).

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1 Introduction

By combining a result of McNaughton and Papert [10] with Schützenberger’s characterization of star-free languages [14], a given language over finite words is definable in first-order logic if and only if its syntactic monoid is finite and aperiodic. The implication from left to right can be shown using Ehrenfeucht-Fraïssé games, see e.g. [15]. A similar result for two-variable first-order logic $\mathbf{FO}^2$ was obtained by Thérien and Wilke [17]: A language is definable in $\mathbf{FO}^2$ if and only if its syntactic monoid belongs to the variety $\mathbf{DA}$. Both the variety $\mathbf{DA}$ and the class of finite aperiodic monoids can be defined using identities of omega-terms. Roughly speaking, omega-terms are words equipped with an additional operation, the $\omega$-power. If $M$ is a finite monoid, then there exists a positive integer $\omega_M$ such that $u^{\omega_M} = (u^{\omega_M})^2$ for all $u \in M$. We call $u^{\omega_M}$ the idempotent generated by $u$. Every mapping $h : \Lambda \to M$ uniquely extends to omega-terms over $\Lambda$ by setting $h(st) = h(s)h(t)$ and $h(s^\omega) = h(s)^{\omega_M}$. Now, $M$ satisfies an identity $u = v$ of omega-terms $u$ and $v$ over the alphabet $\Lambda$ if for every mapping $h : \Lambda \to M$ we have $h(s) = h(t)$. A finite monoid is aperiodic if and only if it satisfies $x^{\omega} = x^{\omega}x$, and it is in $\mathbf{DA}$ if and only if it satisfies $(xyz)^\omega y(xyz)^\omega = (xyz)^\omega$, see e.g. [13]. Showing that some first-order fragment $\mathcal{F}$ satisfies an identity $u = v$ of omega-terms $u, v$ usually works as follows. Suppose $\mathcal{F}$ does not satisfy $u = v$. Then there exists a formula $\varphi \in \mathcal{F}$ such that the syntactic monoid of $L(\varphi)$ does not
satisfy $u = v$. The depth $n$ of the formula $\varphi$ defines an $n$-round Ehrenfeucht-Fraïssé game on instances of $u$ and $v$ (i.e., on finite words which are obtained by replacing the $\omega$-powers by fixed positive integers depending on $n$). Giving a winning strategy for duplicator yields a contradiction, thereby showing that $\mathcal{F}$ satisfies $u = v$. Usually, playing the game on $u$ and $v$ involves some non-trivial book-keeping since one has to formalize intuitive notions such as positions being near to one another or being close to some border. For first-order logic and for $\text{FO}^2$ the book-keeping is still feasible [15, 4] whereas for other fragments such as the quantifier alternation inside $\text{FO}^2$ this task becomes much more involved (and therefore other techniques are applied [8, 16]).

Instead of defining new instances of a given omega-term depending on the fragment and the number of rounds in the Ehrenfeucht-Fraïssé game, we give a single instance which works for all fragments of first-order logic and any number of rounds. In addition, we allow an infinite number of rounds. The fragments we consider in this paper rely on an abstract notion of a logic fragment as introduced in [7]. We show that a fragment $\mathcal{F}$ satisfies an identity of omega-terms if and only if duplicator has a winning strategy for the Ehrenfeucht-Fraïssé game for $\mathcal{F}$ on the instances of the omega-terms. These instances are labeled linear orders which, in general, are not finite words.

An obvious application of our main result is the simplification of proofs showing that some fragment $\mathcal{F}$ satisfies a given identity of omega-terms. The main reason is that with this new approach one can avoid the book-keeping. It is slightly less straightforward that one can use this approach for solving word problems for omega-terms over varieties of finite monoids. Let $\mathbf{V}$ be a variety of finite monoids. Then the word problem for omega-terms over $\mathbf{V}$ is the following: Given two omega-terms $u$ and $v$, does every monoid in $\mathbf{V}$ satisfy the identity $u = v$? This problem was solved for various varieties, see e.g. [2, 9, 11]. Using our main result, one approach to solving such word problems is as follows. First, find a logical fragment for $\mathbf{V}$. Second, find a winning strategy for duplicator on omega-terms satisfied by this fragment. Third, use this winning strategy for finding the desired decision algorithm. In the case of aperiodic monoids, we use this scheme for improving the decidability result of McCammond [9] by showing that the word problem for omega-terms over aperiodic monoids is solvable in exponential time.

Historically, the greek letter $\omega$ is used for two different things which are frequently used throughout this paper: First, the idempotent power of an element and second, the smallest infinite ordinal. In order to avoid confusion in our presentation, we chose to follow the approach of Perrin and Pin [12] by using $\pi$ instead of $\omega$ to denote idempotent powers. In particular, we will use the exponent $\pi$ in omega-terms which is why we will call them $\pi$-terms in the remainder of this paper.

\section{Preliminaries}

As mentioned above, one of the central notions in this paper are so called $\pi$-terms. In order to make their interpretation by several semantics possible in a uniform way, we follow an algebraic approach. A $\pi$-\textit{algebra} is a structure $(U, \cdot, \pi)$ comprised of a binary operation $\cdot$ and a unary operation $\pi$ on a carrier set $U$. The application of $\cdot$ is usually written as juxtaposition, i.e., $uv = u \cdot v$, and the application of $\pi$ to $u \in U$ as $u^\pi$. A $\pi$-\textit{term} is an arbitrary element of the free $\pi$-algebra $T_{\Lambda}$ generated by $\Lambda$, where $\Lambda$ is a \textbf{countably infinite set which is fixed for the rest of this paper}. We also use this set as a universe for letters (of alphabets).
Monoids as $\pi$-Algebras. Let $M$ be a monoid. For any $k \geq 1$ we extend $M$ to a $\pi$-algebra, called $k$-power algebra on $M$, by defining $u^k = u^k$ for $u \in M$. Suppose that $M$ is finite. An element $u \in M$ is idempotent if $u^2 = u$. We extend $M$ to another $\pi$-algebra, called idempotency algebra on $M$, by defining $u^\pi$ for $u \in M$ to be the unique idempotent element in the set $\{u^k \mid k \geq 1\}$. In fact, there are infinitely many $k \geq 1$, called idempotency exponents of $M$, such that for each $u \in M$ the element $u^k$ is idempotent, i.e., the $k$-power algebra and the idempotency algebra on $M$ coincide. An identity $s = t$ of $\pi$-terms $s, t \in T_\pi$ holds in $M$ if every $\pi$-algebra morphism $h$ from $T_\pi$ into the idempotency algebra on $M$ satisfies $h(s) = h(t)$.

The set of all finite words over an alphabet $A \subseteq \Lambda$ is $A^*$. Let $L \subseteq A^*$ be language over a finite alphabet $A \subseteq \Lambda$. The syntactic congruence of $L$ is the equivalence relation $\equiv_L$ on $A^*$ defined by $u \equiv_L v$ if $xuy \in L$ is equivalent to $xvy \in L$ for all $x, y \in A^*$. In fact, $\equiv_L$ is a monoid congruence on $A^*$. The quotient monoid $M_L = A^*/\equiv_L$ is called syntactic monoid of $L$. It is finite precisely if $L$ is regular. Suppose that $L$ is regular and let $k \geq 1$ be an idempotency exponent of $M_L$. Then the map sending each $u \in A^*$ to its $\equiv_L$-class is a $\pi$-algebra morphism from the $k$-power algebra on $A^*$ onto the idempotency algebra on $M_L$. Thus, any identity $s = t$ of $\pi$-terms $s, t \in T_\pi$ holds in $M_L$ if, and only if, every $\pi$-algebra morphism $h$ from $T_\pi$ into the $k$-power algebra on $A^*$ satisfies $h(s) = h(t)$.

Generalised Words. The third semantic domain we consider is the class of generalised words. A generalised word (over $\Lambda$) is a triple $u = (P_u, \leq_u, \ell_u)$ comprised of a (possibly empty) linear ordering $(P_u, \leq_u)$ being labelled by a map $\ell_u : P_u \to \Lambda$. The set $\text{dom}(u) = P_u$ is the domain of $u$, its elements are called positions of $u$. We write $u(p)$ instead of $\ell_u(p)$ for $p \in P$. The order type $\text{tp}(u)$ of $u$ is the isomorphism type of $(P_u, \leq_u)$. We regard any finite word $w = a_1 \ldots a_n \in A^*$ as a generalised word by defining $\text{dom}(w) = [1, n]$, $\leq_w$ as the natural order on $[1, n]$ and $\ell(w) = a_k$ for $k \in [1, n]$. On that view, generalised words indeed generalise finite words. As of now, we mean “generalised word” when writing just “word”. Two words $u$ and $v$ are isomorphic if there exists an isomorphism $f$ of linear orderings from $(\text{dom}(u), \leq_u)$ to $(\text{dom}(v), \leq_v)$ such that $u(p) = v(f(p))$ for all $p \in \text{dom}(u)$. We identify isomorphic words. We denote the set of all (isomorphism classes of) countable words by $A^{\text{LO}}$. On the view of finite words being words which are finite, we regard $A^*$ as a subset of $A^{\text{LO}}$.

Let $u, v \in A^{\text{LO}}$ be two words. Their concatenation is the word $uv \in A^{\text{LO}}$ defined by $\text{dom}(uv) = \text{dom}(u) \cup \text{dom}(v)$, $\leq_{uv}$ makes all positions of $u$ smaller than those of $v$ and retains the respective orders inside $u$ and inside $v$, and $(uv)(p)$ is $u(p)$ if $p \in \text{dom}(u)$ and $v(p)$ if $p \in \text{dom}(v)$. The set $A^{\text{LO}}$ with concatenation forms a monoid. On finite words this concatenation coincides with the usual definition and hence $A^*$ is a submonoid of $A^{\text{LO}}$.

It is customary to regard $n \in \mathbb{N}$ also as the order type of the natural linear ordering on $[1, n]$. We extend the notion of the $n$-power algebra on $A^{\text{LO}}$ to arbitrary countable order types $\tau$ as follows. Let $(T, \leq_T)$ be a linear ordering of isomorphism type $\tau$. The $\tau$-power of any word $u \in A^{\text{LO}}$ is the word $u^\tau \in A^{\text{LO}}$ defined by $\text{dom}(u^\tau) = \text{dom}(u) \times T$, $(p, t) \leq_{u^\tau} (p', t')$ if $t \leq_T t'$ or if $t = t'$ and $p \leq_u p'$, and $(u^\tau)(p, t) = u(p)$. We extend the monoid $A^{\text{LO}}$ to a $\pi$-algebra, called $\tau$-power algebra on $A^{\text{LO}}$, by defining $u^\pi = u^\tau$ for $u \in A^{\text{LO}}$. We denote by $[\cdot]_\tau$ the unique $\pi$-algebra morphism from $T_\pi$ into this $\pi$-algebra mapping each $a \in \Lambda$ to the word consisting of a single position which is labelled by $a$. Finally, notice that there are two definitions of the $n$-power algebra on $A^{\text{LO}}$ around, but actually they coincide.
Logic over Words. For the rest of this paper, we fix a countably infinite set $\mathbb{V}$ of (first-order) variables $x, y, z, \ldots$. The syntax of first-order logic over words is given by

$$
\varphi ::= \top | \bot | \text{empty} | x = y | \lambda(x) = a | x < y | x \leq y | \text{suc}(x, y) | \text{min}(x) | \text{max}(x) |
\neg \varphi | \varphi \lor \varphi | \varphi \land \varphi | \exists x \varphi | \forall x \varphi,
$$

where $x, y \in \mathbb{V}$ and $a \in \Lambda$. The set of all formulae is denoted by $\text{FO}$. The free variables $\text{IV}(\varphi)$ of a formula $\varphi \in \text{FO}$ are defined as usual. A sentence is a formula $\varphi$ with $\text{IV}(\varphi) = \emptyset$.

We only give a brief sketch of the semantics of formulae. Let $X \subseteq \mathbb{V}$ be a finite set of variables. An $X$-valuation on $u$ is a pair $\langle u, \alpha \rangle$ consisting of a word $u \in \Lambda^X$ and a map $\alpha : X \to \text{dom}(u)$. It is a model of a formula $\varphi \in \text{FO}$ with $\text{IV}(\varphi) \subseteq X$, in symbols $\langle u, \alpha \rangle \models \varphi$, if $u$ satisfies the formula $\varphi$ under the following assumptions:

- variables range over positions of $u$ and free variables are interpreted according to $\alpha$,
- $\top$ is always satisfied, $\bot$ never, and empty only in case $\text{dom}(u) = \emptyset$,
- the function symbol $\lambda$ is interpreted by the labelling map $\ell_u : \text{dom}(u) \to \Lambda$ and
- the predicates $<, \leq, \text{suc}, \text{min}$ and $\text{max}$ are evaluated in the linear ordering $(\text{dom}(u), \leq_u)$,

where $\text{suc}(x, y)$ means that $y$ is the immediate successor of $x$.

We identify any word $u \in \Lambda^X$ with the only $\emptyset$-valuation on $u$, namely $\langle u, \emptyset \rangle$ with $\emptyset$ also denoting the empty map. Thus, for sentences $\varphi$ the meaning of $u \models \varphi$ is well-defined.

Let $A \subseteq \Lambda$ be a finite alphabet and $\varphi \in \text{FO}$ a sentence. Due to Büchi’s result, the language over $A$ defined by $\varphi$, namely $L_A(\varphi) = \{ w \in A^* \mid w \models \varphi \}$, is regular. A language $L \subseteq A^*$ is definable in a class $\mathcal{F} \subseteq \text{FO}$ of formulae if there exists a sentence $\varphi \in \mathcal{F}$ such that $L = L_A(\varphi)$.

Fragments. We reintroduce (a slight variation of) the notion of a fragment as a class of formulae obeying natural syntactic closure properties [7]. A context is a formula $\mu$ with a unique occurrence of an additional constant predicate $\circ$ which is intended to be a placeholder for another formula $\varphi \in \text{FO}$. The result of replacing $\circ$ in $\mu$ by $\varphi$ is denoted by $\mu(\varphi)$.

Unfortunately, the notion of a fragment as defined in [7, Definition 1] is slightly too weak for our purposes. We require one more natural syntactic closure property, namely condition 4, in Definition 2.1 below. Condition 6. is missing in the exposition in [7]. Nevertheless, since we only add requirements, every fragment in our sense is still a fragment in the sense of [7].

Definition 2.1. A fragment is a non-empty set of formulae $\mathcal{F} \subseteq \text{FO}$ such that for all contexts $\mu$, formulae $\varphi, \psi \in \text{FO}$, $a \in \Lambda$ and $x, y \in \mathbb{V}$ the following conditions are satisfied:

1. If $\mu(\varphi) \in \mathcal{F}$, then $\mu(\top) \in \mathcal{F}$, $\mu(\bot) \in \mathcal{F}$, and $\mu(\lambda(x) = a) \in \mathcal{F}$.
2. $\mu(\varphi \lor \psi) \in \mathcal{F}$ if, and only if, $\mu(\varphi) \in \mathcal{F}$ and $\mu(\psi) \in \mathcal{F}$.
3. $\mu(\varphi \land \psi) \in \mathcal{F}$ if, and only if, $\mu(\varphi) \in \mathcal{F}$ and $\mu(\psi) \in \mathcal{F}$.
4. If $\mu(\neg \varphi) \in \mathcal{F}$, then $\mu(\varphi) \in \mathcal{F}$.
5. If $\mu(\exists x \varphi) \in \mathcal{F}$ and $x \notin \text{IV}(\varphi)$, then $\mu(\varphi) \in \mathcal{F}$.
6. If $\mu(\forall x \varphi) \in \mathcal{F}$ and $x \notin \text{IV}(\varphi)$, then $\mu(\varphi) \in \mathcal{F}$.

It is closed under negation if the following condition is satisfied:

7. If $\varphi \in \mathcal{F}$, then $\neg \varphi \in \mathcal{F}$.

It is order-stable if the following condition is satisfied:

8. $\mu(x < y) \in \mathcal{F}$ if, and only if, $\mu(x \leq y) \in \mathcal{F}$.

It is suc-stable if the following two conditions are satisfied:

9. If $\mu(\text{suc}(x, y)) \in \mathcal{F}$, then $\mu(x = y) \in \mathcal{F}$, $\mu(\text{max}(x)) \in \mathcal{F}$ and $\mu(\text{min}(y)) \in \mathcal{F}$.
10. If $\mu(\text{min}(x)) \in \mathcal{F}$ or $\mu(\text{max}(x)) \in \mathcal{F}$, then $\mu(\text{empty}) \in \mathcal{F}$.
Duplicator chooses a position $Qx$  

<table>
<thead>
<tr>
<th>1. spoiler chooses $Qx$</th>
<th>2. spoiler chooses $q$ in $Qx$</th>
<th>3. duplicator chooses $r$ in $Qx$</th>
<th>4. resulting situation $S[Qx,q,r]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Qx = \exists x$</td>
<td>$\text{dom}(u)$</td>
<td>$\text{dom}(v)$</td>
<td>$(G/\exists x, \langle u, \alpha[x/q], \langle v, \beta[x/r]\rangle)$</td>
</tr>
<tr>
<td>$Qx = \forall x$</td>
<td>$\text{dom}(v)$</td>
<td>$\text{dom}(u)$</td>
<td>$(G/\forall x, \langle u, \alpha[x/r], \langle v, \beta[x/q]\rangle)$</td>
</tr>
<tr>
<td>$Qx = \neg \exists x$</td>
<td>$\text{dom}(v)$</td>
<td>$\text{dom}(u)$</td>
<td>$(G/\neg \exists x, \langle v, \beta[x/q], \langle u, \alpha[x/r]\rangle)$</td>
</tr>
<tr>
<td>$Qx = \neg \forall x$</td>
<td>$\text{dom}(u)$</td>
<td>$\text{dom}(v)$</td>
<td>$(G/\neg \forall x, \langle v, \beta[x/r], \langle u, \alpha[x/q]\rangle)$</td>
</tr>
</tbody>
</table>

Table 1 Master plan of a single round of the $\mathcal{F}$-game in situation $S = (G, \langle u, \alpha \rangle, \langle v, \beta \rangle)$.

Examples for fragments in this sense include all classes of formulae which are obtained from full first-order logic $\mathsf{FO}$ by limiting the quantifier rank (e.g., $\mathsf{FO}_n$), the number of quantifier alternations (e.g., $\Sigma_n$ and $\Pi_n$), the number of quantified variables (e.g., $\mathsf{FO}^m$), the available predicates (e.g., $\mathsf{FO}[\lt]$) or combinations of those.

The quantifier rank $\text{qr}(\varphi)$ of a formula $\varphi \in \mathsf{FO}$ is defined as usual. A fragment $\mathcal{F}$ has bounded quantifier rank if there is an $n \in \mathbb{N}$ such that $\text{qr}(\varphi) \leq n$ for all $\varphi \in \mathcal{F}$. For any $n \in \mathbb{N}$ and every fragment $\mathcal{F}$ the set $\mathcal{F}_n = \{ \varphi \in \mathcal{F} \mid \text{qr}(\varphi) \leq n \}$ is a fragment of bounded quantifier rank. Moreover, the fragment $\mathcal{F}_n$ is order-stable (respectively suc-stable) in case $\mathcal{F}$ has the according property.

### 3 Ehrenfeucht-Fraïssé Games for Arbitrary Fragments

In this section, we introduce an Ehrenfeucht-Fraïssé game for arbitrary fragments of first-order logic on generalised words and develop its basic theory. Before we can describe this game, we need to define some notation. In the following, we call the “negated quantifiers” $\neg \exists$ and $\neg \forall$ also quantifiers. The set of all quantifiers (in this sense) is $Q = \{ \exists, \forall, \neg \exists, \neg \forall \}$. For a quantifier $Q \in Q$ and a variable $x \in \mathbb{V}$, the reduct of $\mathcal{F}$ by $Qx$ is the set

$$\mathcal{F}/Qx = \{ \varphi \in \mathsf{FO} \mid Qx \varphi \in \mathcal{F} \}.$$  

Whenever this set is not empty, it is a fragment as well.

Now, let $\mathcal{F}$ be a fragment and $u, v$ two words over $\Lambda$. We are about to describe the $\mathcal{F}$-game on $(u, v)$. A situation of this game is a triple $S = (G, \langle u, \alpha \rangle, \langle v, \beta \rangle)$ comprised of a non-empty, iterated reduct $G$ of $\mathcal{F}$ and $\mathbb{X}$-valuations $\langle u, \alpha \rangle$ and $\langle v, \beta \rangle$ on $u$ and $v$ for the same arbitrary finite subset $\mathbb{X} \subseteq \mathbb{V}$. To emphasise the set $\mathbb{X}$, we also speak of an $\mathbb{X}$-situation. The game starts in the $\emptyset$-situation ($\mathcal{F}, u, v$) and goes on for arbitrarily many—possibly infinitely many—rounds. Assuming that the game is currently in situation $S = (G, \langle u, \alpha \rangle, \langle v, \beta \rangle)$, a single round proceeds as follows (cf. Table 1 for a summary of this procedure):

1. Spoiler chooses a quantifier $Q \in Q$ and a variable $x \in \mathbb{V}$ such that $G/Qx \neq \emptyset$.
2. Spoiler chooses a position $q$ (like “quest”) in the domain of $u$ if $Q \in \{ \exists, \neg \exists \}$ or in the domain of $v$ if $Q \in \{ \forall, \neg \forall \}$.
3. Duplicator chooses a position $r$ (like “reply”) in the domain of the other word.
4. The resulting situation $S[Qx,q,r]$ consists of the reduct $G/Qx$ and the extension of the $\mathbb{X}$-valuations $\langle u, \alpha \rangle$ and $\langle v, \beta \rangle$ by variable $x$ at positions $q$ and $r$, accordingly. Whenever $Q$ is a negated quantifier, the role of the two extended valuations is additionally interchanged. Whenever a player cannot perform a choice because $G$ contains no more quantified formulae or the domain of the according word is empty, the game immediately stops and the other player wins. Spoiler’s objective in the game is to eventually reach an $\mathbb{X}$-situation $(G, \langle u, \alpha \rangle, \langle v, \beta \rangle)$ which satisfies his winning condition that there exists a literal $\varphi \in G$ with
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\( \text{IV}(\varphi) \subseteq X \) and \((u, \alpha) \models \varphi \) but \((v, \beta) \not\models \varphi \). Duplicator’s goal is simply to prevent spoiler from winning. In particular, duplicator wins all games that go on forever. Due to this circumstance, the \( F \)-game is determined, i.e., either spoiler or duplicator has a winning strategy on \((u, v)\).

\( \bowtie \) Remark. The \( F \)-game is quite asymmetric since spoiler is not allowed to choose before his first move whether he wants to play on \((u, v)\) or on \((v, u)\). This may lead to the situation that he has a winning strategy on \((u, v)\) but not on \((v, u)\) or vice versa. This asymmetry is owed to the circumstance that \( F \) might not be closed under negation. As soon as \( F \) is assumed to be closed under negation this phenomenon disappears and spoiler has a winning strategy on \((u, v)\) if, and only if, he has a winning strategy on \((v, u)\). However, since we do not make this assumption in general, most of the results look either asymmetric or “symmetrified”.

If the quantifier rank of a fragment \( F \) is bounded by \( n \in \mathbb{N} \), the \( F \)-game lasts at most \( n \) rounds. In particular, for any fragment \( F \) the \( F_n \)-game can be regarded as an \( n \)-round version of the \( F \)-game. For instance, the \( \text{FO}_n \)-game resembles the classical \( n \)-round Ehrenfeucht-Fraïssé game. The following theorem is an adaption of the Ehrenfeucht-Fraïssé theorem to the \( F \)-game for fragments of bounded quantifier rank.

\( \bowtie \) Theorem 3.1. Let \( F \) be a fragment of bounded quantifier rank. For all words \( u, v \in \Lambda^{\text{LO}} \) the following are equivalent:

1. \( u \models \varphi \) implies \( v \models \varphi \) for all sentences \( \varphi \in F \) and
2. duplicator has a winning strategy in the \( F \)-game on \((u, v)\).

A proof of this theorem can easily be achieved along the lines of a proof of the classical version, cf. [6]. In fact, such a proof reveals that the implication “2. \( \Rightarrow \) 1.” even holds if the quantifier rank of \( F \) is not bounded. In contrast, the implication “1. \( \Rightarrow \) 2.” substantially relies the boundedness of the quantifier depth of \( F \). For instance, duplicator has a winning strategy in the \( \text{FO}_n \)-game on \((a^\xi, a^{\xi+\zeta})\) for each \( n \in \mathbb{N} \) and hence \( a^\xi \models \varphi \) implies \( a^{\xi+\zeta} \models \varphi \) for all sentences \( \varphi \in \text{FO} \), but spoiler has a winning strategy in the \( \text{FO} \)-game on \((a^\xi, a^{\xi+\zeta})\).

The objective of the remainder of this section is to identify additional requirements on \( F \) and/or \( u, v \) such that the boundedness of the quantifier rank can be omitted. It turns out that the property introduced in Definition 3.2 below in combination with suc-stability of the fragment is sufficient for this purpose and still allows for the applications in Section 4. The order types of the sets \( \mathbb{N}, \mathbb{Z}, \mathbb{Q} \) and \( \mathbb{Z}_{<0} \) ordered naturally are denoted by \( \omega, \zeta, \eta \) and \( \omega^* \), respectively. Then \( \omega + \zeta \cdot \eta + \omega^* \) is the order type of the word \( a^\omega (a^\xi)^\eta a^{\omega^*} \), where \( a \in \Lambda \).

\( \bowtie \) Definition 3.2. Let \( \varpi = \omega + \zeta \cdot \eta + \omega^* \). A word \( u \in \Lambda^{\text{LO}} \) is \( \varpi \)-rational if it can be constructed from the finite words in \( \Lambda^{\text{LO}} \) using the operations of concatenation and \( \varpi \)-power only or, equivalently, if \( u = [t]_\varpi \) for some \( \pi \)-term \( t \in T_\Lambda \).

\( \bowtie \) Theorem 3.3. Let \( F \) be a suc-stable fragment. For all \( \varpi \)-rational words \( u, v \in \Lambda^{\text{LO}} \) the following are equivalent:

1. \( u \models \varphi \) implies \( v \models \varphi \) for all sentences \( \varphi \in F \) and
2. duplicator has a winning strategy in the \( F \)-game on \((u, v)\).

As already mentioned, the implication “2. \( \Rightarrow \) 1.” can be shown using the very same proof as for the according implication of Theorem 3.1. The key idea behind proving the implication “1. \( \Rightarrow \) 2.” is as follows: Theorem 3.1 provides us for each \( n \in \mathbb{N} \) with a winning strategy for duplicator in the \( F_n \)-game on \((u, v)\). A winning strategy in the \( F \)-game is obtained by “taking the limit” of all those strategies and “fusing them together” into one strategy.
This limit process relies of the \( \varpi \)-rationality of the underlying words and is formalised by Lemma 3.6 below. The main ingredients of its proof is Proposition 3.4.

In order to keep notation concise, we abbreviate the circumstance that duplicator has a winning strategy in a situation \( S = (\mathcal{F}, \langle u, \alpha \rangle, \langle v, \beta \rangle) \) by \( \langle u, \alpha \rangle \preceq_S (v, \beta) \). Since the \( \mathcal{F} \)-game is determined, \( \langle u, \alpha \rangle \not\preceq_S (v, \beta) \) hence means that spoiler has a winning strategy in \( S \). The relation \( \not\preceq_S \) is reflexive and transitive, i.e., a preorder on the set of all situations. It induces an equivalence \( \equiv_S \) defined by \( \langle u, \alpha \rangle \equiv_S (v, \beta) \) if \( \langle u, \alpha \rangle \not\preceq_S (v, \beta) \) and \( (v, \beta) \not\preceq_S \langle u, \alpha \rangle \).

\[ \text{ Proposition 3.4. Let } \mathcal{F} \text{ be a suc-stable fragment, } k \in \mathbb{N} \text{ and } \langle u_i, \alpha_i \rangle, \langle v_i, \beta_i \rangle \text{ } \mathcal{X}_i \text{-valuations with mutually disjoint } \mathcal{X}_i \text{ for } i \in [1, k]. \text{ If } \langle u_i, \alpha_i \rangle \not\preceq_{\mathcal{F}} (v_i, \beta_i) \text{ for each } i \in [1, k], \text{ then } \langle u_1 \cdots u_k, \alpha_1 \cup \cdots \cup \alpha_k \rangle \not\preceq_{\mathcal{F}} (v_1 \cdots v_k, \beta_1 \cup \cdots \cup \beta_k) . \]

\[ \text{ Lemma 3.5. Let } \mathcal{F} \text{ be a suc-stable fragment whose quantifier rank is bounded by } n \in \mathbb{N} \text{ and } u, v \in \Lambda_{\mathcal{L}O}. \text{ If } u \not\preceq_{\mathcal{F}} v, \text{ then } u^n \not\preceq_{\mathcal{F}} v^n \text{ and } u^n \not\preceq_{\mathcal{F}} v^m \text{ for all } m \geq 2^n - 1 \text{ but } m = 0 . \]

The following lemma formalises the limit process mentioned above.

\[ \text{ Lemma 3.6. Let } \mathcal{F} \text{ be a suc-stable fragment, } x \in \mathbb{V} \text{ and } \langle u, \alpha \rangle \text{ an } \mathcal{X} \text{-valuation on a } \varpi \text{-rational word } u \in \Lambda_{\mathcal{L}O}. \text{ For every infinite sequence } (q_i)_{i \in \mathbb{N}} \in \text{dom}(u)^{\mathbb{N}} \text{ there exists a position } q \in \text{dom}(u) \text{ such that for all } n \in \mathbb{N} \text{ there are arbitrarily large } i \in \mathbb{N} \text{ with } \langle u, \alpha[x/q_i] \rangle \not\preceq_{\mathcal{F}} (u, \alpha[x/q_i]). \]

\[ \text{ Proof. To simplify notation, we call a position } q \text{ with the property above a } \langle u, \alpha \rangle \text{-limit point of the sequence } (q_i)_{i \in \mathbb{N}} \text{ (w.r.t. to } \mathcal{F} \text{ and } x) . \text{ Using this terminology, we have to show that every sequence } (q_i)_{i \in \mathbb{N}} \in \text{dom}(u)^{\mathbb{N}} \text{ possesses a } \langle u, \alpha \rangle \text{-limit point. Since neither } \alpha[x/q_i] \text{ nor } \alpha[x/q_i] \text{ would depend on } \alpha(x), \text{ we may simply assume that } x \notin \mathcal{X} . \text{ We proceed by induction on the } \varpi \text{-rational construction of } u. \]

\[ \text{ Base case: } u \text{ is finite. Since } \text{dom}(u) \text{ is finite, there exists a } q \in \text{dom}(u) \text{ such that } q = q_i \text{ for infinitely many } i \in \mathbb{N} . \text{ Thus, } q \text{ is a } \langle u, \alpha \rangle \text{-limit point of } (q_i)_{i \in \mathbb{N}} . \]

\[ \text{ Inductive step 1: } u = v_1 \cdots v_k \text{ with } \varpi \text{-rational words } v_1, \ldots, v_k . \text{ The valuation } \langle u, \alpha \rangle \text{ splits intovaluations } \langle v_i, \beta_i \rangle \text{ for } i \in [1, k] \text{ such that } \alpha = \beta_1 \cup \cdots \cup \beta_k . \text{ There is an } \ell \in [1, k] \text{ such that } q_i \in \text{dom}(v_i) \text{ for infinitely many } i \in \mathbb{N} . \text{ Let } I \text{ be the set of these } i . \text{ By the induction hypothesis, there is a } \langle v_i, \beta_i \rangle \text{-limit point } q \in \text{dom}(v_i) \text{ of the subsequence } (q_i)_{i \in I} . \text{ Proposition 3.4 implies that } q \text{ is also a } \langle u, \alpha \rangle \text{-limit point of } (q_i)_{i \in \mathbb{N}} . \]

We split the inductive step for \( \varpi \)-power in two parts, one for \( \mathcal{X} = \emptyset \) and another for \( \mathcal{X} \neq \emptyset \).

\[ \text{ Inductive step 2: } u = v^m \text{ with a } \varpi \text{-rational } v \text{ and } \mathcal{X} = \emptyset . \text{ Let } (P, \leq_P) \text{ be a linear ordering of isomorphism type } \varpi \text{ such that } \text{dom}(u) = \text{dom}(v) \times P . \text{ For each } i \in \mathbb{N} \text{ we write } q_i = (s_i, p_i) . \text{ For every } p \in P \text{ let } \varpi_p = \varpi_{\mathcal{F}} \text{ and } \varpi_p \text{ be the order types of the suborders of } (P, \leq_P) \text{ induced by the open intervals } (-\infty, p) \text{ and } (p, +\infty), \text{ respectively. Then } \varpi = \varpi_p + 1 + \varpi_p . \text{ Due to the nature of } \varpi, \text{ each of } \varpi_p \text{ and } \varpi_p \text{ is either finite or equals } \varpi . \text{ However, the case that } \varpi_p \text{ and } \varpi_p \text{ both are finite at the same time cannot occur. Accordingly, we distinguish three cases:} \]

\[ \text{ Case 1: } \varpi_p = \varpi_p = \varpi \text{ for infinitely many } i \in \mathbb{N} . \text{ Let } I \text{ be the set of these } i . \text{ By the induction hypothesis, there exists a } \langle v, \theta \rangle \text{-limit point } s \in \text{dom}(v) \text{ of the subsequence } (s_i)_{i \in I} . \text{ We pick some } j \in I . \text{ Proposition 3.4 reveals that } q = (s_j, p_j) \text{ is a } \langle u, \alpha \rangle \text{-limit point of } (q_i)_{i \in \mathbb{N}} . \]

\[ \text{ Case 2: } \varpi_p \text{ is finite and } \varpi_p = \varpi \text{ for infinitely many } i \in \mathbb{N} . \text{ Let } I \text{ be the set of these } i . \text{ If there is an order type which occurs infinitely often among the } \varpi_p \text{ with } i \in I, \text{ the same} \]
argumentation as in Case 1 applies. Henceforth, we assume that such an order type does not exist. By the induction hypothesis, the subsequence \((s_i)_{i \in I}\) possesses a \((v, 0)\)-limit point \(s \in \text{dom}(v)\). Let \(p \in P\) be arbitrary with \(\tau_p = \tau_p = \omega\). We show that \(q = (s, p)\) is a \(\langle u, \alpha \rangle\)-limit point of \((\varphi_i)_{i \in \mathbb{N}}\).

Let \(n \in \mathbb{N}\). Due to the choice of \(I\) and \(s\), there are arbitrarily large \(i \in I\) such that \(\tau_{pi}\) is of size at least \(2^n - 1\) and \(\langle v, 0|x/s| \rangle \subseteq F_n \langle v, 0|x/s| \rangle\). Proposition 3.5 then implies \(v^{\omega n} \subseteq F_n v^\omega\). Since also \(v^{\omega n} \subseteq F_n v^\omega\), Proposition 3.4 yields \(\langle u, 0|x/q| \rangle \subseteq F_n \langle u, 0|x/q| \rangle\).

**Case 3:** \(\tau_{pi} = \omega\) and \(\tau_{pi}\) is finite for infinitely many \(i \in \mathbb{N}\). Symmetric to Case 2.

**Inductive step 3:** \(u = v^\omega\) with a \(\omega\)-rational \(v\) and \(X \neq \emptyset\). Let \((P, \leq P)\) be as above. Recall that \(X\) is supposed to be finite. Let \(\tilde{p}_1 < P \cdots < P \tilde{p}_k\) be an enumeration of all \(p \in P\) for which there exists a \(y \in X\) such that \(\alpha(y) \in \text{dom}(v) \times \{p\}\). We consider the open intervals \(P_0 = (-\infty, \tilde{p}_1), P_\ell = (\tilde{p}_\ell, \tilde{p}_{\ell+1})\) for \(\ell \in [1, k - 1]\), and \(P_k = (\tilde{p}_k, +\infty)\) in \((P, \leq P)\). For \(\ell \in [0, k]\) we let \(\tau_\ell\) be the order type of the suborder induced by \(P_\ell\). Then \(\omega = \tau_0 + 1 + \tau_1 + \cdots + 1 + \tau_k\) and hence \(u = v^{\tau_0} v^{\tau_1} v \cdots v^{\tau_k}\). Due to the nature of \(\omega\), each \(\tau_\ell\) is either finite or equals \(\omega\). Since for every finite \(\tau_\ell\) the word \(v^{\tau_\ell}\) is the concatenation of \(\tau_\ell\) copies of \(v\), the factorisation of \(u\) above is an alternative \(\omega\)-rational factorisation of \(u\). This construction has the additional property that \(\alpha\) does not map into the \(\omega\)-powers \(v^\omega\).

Thus, the induction hypothesis and the inductive steps 1 and 2 above yield the claim.  

Now, we are prepared to prove the remaining implication of Theorem 3.3.

**Proof of Theorem 3.3. “1. \(\Rightarrow\) 2.”** We show that duplicator can maintain the invariant of staying in situations which are good for her. A situation \((G, \langle u, \alpha \rangle, \langle v, \beta \rangle)\) of the \(F\)-game on \((u, v)\) is considered to be good for duplicator if \(\langle u, \alpha \rangle \subseteq G_n \langle v, \beta \rangle\) for every \(n \in \mathbb{N}\). Statement 1. and Theorem 3.1 imply that the starting situation \((F, u, v)\) is good. Moreover, good situations do not meet spoiler’s winning condition as they particularly satisfy \(\langle u, \alpha \rangle \subseteq G_\omega \langle v, \beta \rangle\). Consequently, it suffices to provide a strategy for duplicator which never leaves the set of good situations since such a strategy is a winning strategy.

For this purpose, suppose that spoiler chooses quantifier \(Qx\) and quest \(q\) in a good situation \((G, \langle u, \alpha \rangle, \langle v, \beta \rangle)\). We only demonstrate the case \(Q = \exists\), where \(q \in \text{dom}(u)\).

For every \(n \in \mathbb{N}\) we have \(\langle u, \alpha \rangle \subseteq G_{n+1} \langle v, \beta \rangle\) and hence there is an \(r_n \in \text{dom}(v)\) such that \(\langle u, \alpha[x/r] \rangle \subseteq G_{n+1/3x} \langle v, \beta[x/r_n] \rangle\). Since \(G_{n+1/3x} = G_{n/3x}\), this is the same as \(\langle u, \alpha[x/r] \rangle \subseteq G_{n/3x} \langle v, \beta[x/r_n] \rangle\). Due to Lemma 3.6, there exists an \(r \in \text{dom}(v)\) such that for every \(n \in \mathbb{N}\) there are infinitely many \(i \in \mathbb{N}\) with \(\langle v, \alpha[x/r_i] \rangle \subseteq G_{n/3x} \langle v, \alpha[x/r_i] \rangle\).

We show that the situation \(S[\exists x, q, r] = (G_{n/3x}, \langle u, \alpha[x/q] \rangle, \langle v, \beta[x/r] \rangle)\) is good again.

Let \(n \in \mathbb{N}\). Due to the choice of \(r\), there is an \(i \geq n\) with \(\langle v, \alpha[x/r_i] \rangle \subseteq G_{n/3x} \langle v, \alpha[x/r_i] \rangle\). The position \(r_i\) was chosen such that \(\langle u, \alpha[x/r] \rangle \subseteq G_{n/3x} \langle v, \beta[x/r_i] \rangle\). Since \(n \leq i\), this implies \(\langle u, \alpha[x/r] \rangle \subseteq G_{n/3x} \langle v, \beta[x/r_i] \rangle\) and in turn \(\langle u, \alpha[x/r] \rangle \subseteq G_{n/3x} \langle v, \beta[x/r_i] \rangle\).  

## 4 Ehrenfeucht-Fraïssé Games on Identities

A recurring problem in studying the expressive powers of fragments of first-order logic on finite words is to show that a certain identity of \(\pi\)-terms holds in the syntactic monoid/semigroup of every language definable in the fragment under consideration. Theorems 4.1 and 4.2 below can remarkably simplify this task, as demonstrated at the end of this section. In fact, the two theorems are just slight variations of one another and the sole reason for having two theorems is that the \(\text{succ}\)-predicate does not play well with syntactic monoids but only with syntactic semigroups.
Theorem 4.1. Let $F$ be an order-stable fragment not containing the predicates suc, min, max and empty. For all $\pi$-terms $s, t \in T_\Lambda$ the following are equivalent:
1. The identity $s = t$ holds in the syntactic monoid of every language definable in $F$.
2. Duplicator has winning strategies in the $F$-games on $([s]_\Lambda, [t]_\Lambda)$ and $([t]_\Lambda, [s]_\Lambda)$.

Theorem 4.2. Let $F$ be a suc-stable and order-stable fragment. For all $\pi$-terms $s, t \in T_\Lambda$ the following are equivalent:
1. The identity $s = t$ holds in the syntactic semigroup of every language definable in $F$ over non-empty words.
2. Duplicator has winning strategies in the $F$-games on $([s]_\Lambda, [t]_\Lambda)$ and $([t]_\Lambda, [s]_\Lambda)$.

The main ingredients of the proofs of both theorems are Theorem 3.3 and [7, Proposition 2] which is restated as Proposition 4.3 below.

Proposition 4.3. Let $F$ be a fragment, $A, B \subseteq \Lambda$ finite alphabets and $h$ a monoid morphism from $A^*$ into $B^*$. Suppose the following:
1. If $F$ contains the predicate $\leq$ or $<$, then $F$ is order-stable or $h(A) \subseteq B \cup \{\varepsilon\}$.
2. If $F$ contains the predicate suc, min, max or empty, then $\varepsilon \notin h(A)$.

Then for every language $L \subseteq A^*$ definable in $F$, the language $h^{-1}(L)$ is also definable in $F$.

Applying this proposition to $F$-games yields that monoid morphisms satisfying the two conditions above preserve the existence of winning strategies for duplicator.

Corollary 4.4. Let $F, A, B$ and $h$ be as in Proposition 4.3 and suppose they also satisfy conditions 1. and 2.. Then $u \succeq_F v$ implies $h(u) \succeq_F h(v)$ for all $u, v \in A^*$.

Proof. Let $u, v \in A^*$ with $u \succeq_F v$. Since finite words are $\omega$-rational and due to Theorem 3.3, it suffices to show that $h(u) \models \varphi$ implies $h(v) \models \varphi$ for all sentences $\varphi \in F$. Consider a sentence $\varphi \in F$. By Proposition 4.3, there is a sentence $\psi \in F$ such that $L_A(\psi) = h^{-1}(L_B(\varphi))$. Altogether, $h(u) \models \varphi$ implies $u \models \psi$ and since $u \succeq_F v$ this implies $v \models \psi$ which in turn implies $h(v) \models \varphi$.

The following corollary is an immediate consequence of Proposition 3.4 and Lemma 3.5.

Corollary 4.5. Let $F$ be a suc-stable fragment whose quantifier rank is bounded by $n \in \mathbb{N}$ and let $t \in T_\Lambda$ be a $\pi$-term. Then $[t]_\Lambda \approx_F [t]_m$ for all $m \geq 2^n - 1$ but $m = 0$.

The previous results allow us to show Theorems 4.1 and 4.2. However, since their proofs are as similar as their statements, we only demonstrate the first one.

Proof of Theorem 4.1. Let $A \subseteq \Lambda$ be the finite set containing all $u \in \Lambda$ appearing in $s$ or $t$.

We show both implications separately.

1. $\Rightarrow$ 2.". By Theorem 3.3, it suffices to show for every sentence $\varphi \in F$ that $[s]_\Lambda \models \varphi$ iff $[t]_\Lambda \models \varphi$. Consider a sentence $\varphi \in F$ and put $n = qr(\varphi)$. We put $L = L_A(\varphi)$ and let $k \geq 2^n - 1$ be an idempotence exponent of $M_L$. We consider an arbitrary $\pi$-algebra morphism $h$ from $T_\Lambda$ into the $k$-power algebra on $A^*$ with $h(a) = a$ for each $a \in A$. Because $s = t$ holds in $M_L$, we have $h(s) \equiv_L h(t)$. Since $h(s) = [s]_k$ as well as $h(t) = [t]_k$ and by Corollary 4.5, we obtain $h(s) \approx_F [s]_\Lambda$ and $h(t) \approx_F [t]_\Lambda$. Altogether, we conclude that $[s]_\Lambda \models \varphi$ iff $h(s) \models \varphi$ iff $h(t) \models \varphi$ iff $[t]_\Lambda \models \varphi$.

2. $\Rightarrow$ 1.". Let $B \subseteq \Lambda$ be a finite alphabet and $L \subseteq B^*$ a language defined by a sentence $\varphi \in F$. Let $n = qr(\varphi)$ and $k \geq 2^n - 1$ be an idempotence exponent of $M_L$. We have to show that every $\pi$-algebra morphism $g$ from $T_\Lambda$ into the $k$-power algebra on $B^*$ satisfies
g(s) ≡_L g(t). Consider such a morphism g and let h be the unique monoid morphism from A* into B* such that h(a) = g(a) for each a ∈ A. Then g(s) = h([s]_k) and g(t) = h([t]_k). Corollary 4.5 and the assumption \([s]_\equiv \approx_{\varphi} [t]_\equiv\) yield \([s]_k \approx_{\varphi_n} \approx_{\varphi_n} [t]_k \approx_{\varphi_n} [t]_k\). We conclude \(g(s) \approx_{\varphi_n} g(t)\) by Corollary 4.4. By Proposition 3.4, we obtain \(ug(s)v \approx_{\varphi_n} ug(t)v\) for all \(u, v \in B^*\). Since \(\varphi \in \varphi_n\), this finally implies \(g(s) \equiv_L g(t)\).

In the remainder of this section, we demonstrate two applications of Theorem 4.1 by providing quite short proofs of two well-known results. A finite monoid \(M\) is called aperiodic if the identity \(a^*a = a^*\) holds in \(M\).

**Corollary 4.6.** The syntactic monoid of every first-order definable language is aperiodic.

*Proof.* Let \(\text{FO}[\lt]\) be the fragment of all formulae not containing the predicates \(\text{suc, min, max}\) and \(\text{empty}\). Since the meanings of these predicates can be expressed in \(\text{FO}[\lt]\), definability in \(\text{FO}\) and \(\text{FO}[\lt]\) coincide. By Theorem 4.1 applied to \(\text{FO}[\lt]\), it suffices to show \([a^*a]_\equiv ≡_{\text{FO}[\lt]} [a^*]_\equiv\). The property \(\tilde{\varpi} + 1 = \varpi\) of the order type \(\varpi\) implies \([a^*a]_\equiv = [a^*]_\equiv\) and the claim follows.

The result of the second demonstration relates definability in \(\text{FO}_2^1[\lt]\) to the class \(\text{DA}\). The fragment \(\text{FO}_2^1[\lt]\) consists of all formulae not containing the predicates \(\text{suc, min, max}\) and \(\text{empty}\) which quantify over two fixed variables \(x_1, x_2 \in \mathbb{V}\) only. The class \(\text{DA}\) consists of all finite monoids in which the identity \((abc)^*b(abc)^* = (abc)^*\) holds. A significant amount of book-keeping is involved when showing that the syntactic monoid of every \(\text{FO}_2^1[\lt]\)-definable language is in \(\text{DA}\) by applying the classical Ehrenfeucht-Fraïssé game approach, see e.g. [4]. On the other hand, the abstract idea of this proof is very simple: Duplicator copies every move near the left and near the right border, and he does not need to care in the center. We now show that this idea can easily be formalized when using Theorem 4.1.

**Corollary 4.7.** The syntactic monoid of any language definable in \(\text{FO}_2^2[\lt]\) is in \(\text{DA}\).

*Proof.* Let \(s = (abc)^*b(abc)^*\) and \(t = (abc)^*\). Again by Theorem 4.1, it suffices to show \([s]_\equiv ≡_{\text{FO}_2^2[\lt]} [t]_\equiv\). With \(u = (abc)^\varpi\) and \(v = (abc)^{\varpi+\omega}\) we obtain

\[
[s]_\equiv = u((abc)^{\varpi+\omega}b(abc)^{\varpi+\omega}v) \quad \text{and} \quad [t]_\equiv = u((abc)^{\varpi+\omega}(abc)^{\varpi+\omega}v).
\]

Since \(\text{FO}_2^2[\lt]\) is closed under negation and due to Proposition 3.4, it further suffices to show that duplicator has a winning strategy in the \(\text{FO}_2^2[\lt]\)-game on

\[
((abc)^{\varpi+\omega}b(abc)^{\varpi+\omega}, (abc)^{\varpi+\omega}(abc)^{\varpi+\omega})
\]

The strategy is to choose a reply that is labelled by the same letter as the request and such that the positions corresponding to \(x_1\) and \(x_2\) are in the same order in both words. This is always possible, since in both words there are always infinitely many positions to the left (respectively to the right) of any position which are labelled by a given letter from \(a, b, c\).
The Word Problem for $\pi$-Terms over Aperiodic Monoids

The word problem for $\pi$-terms over aperiodic monoids was solved by McCammond [9] by computing normal forms. In the process of computing these normal forms the intermediate terms can grow and, to the best of our knowledge, neither the worst-case running time nor the maximal size of the intermediate terms has been estimated (and it seems to be difficult to obtain such results). In this section we give an exponential algorithm for solving the word problem for $\pi$-terms over aperiodic monoids. Our algorithm does not compute normal forms as $\pi$-terms; instead we show that the evaluation under $[\cdot]_\pi$ can be used as a normal form for $\pi$-terms.

\begin{theorem}
Given two $\pi$-terms $s, t \in T_\Lambda$, one can decide whether the identity $s = t$ holds in every aperiodic monoid in time exponential in the sizes of $s$ and $t$.
\end{theorem}

The proof is mainly a reduction to the isomorphism problem for regular words, cf. [3]. These generalised words particularly include all $\pi$-rational words and can be described by expressions similar to $\pi$-terms but using $\omega$-power, $\omega^*$-power and dense shuffle instead of the $\pi$-power. Due to [3, Theorem 79], one can decide in polynomial time whether two such expressions describe isomorphic words. The characterisation underlying the reduction is as follows:

\begin{prop}
For all $\pi$-terms $s, t \in T_\Lambda$ the following are equivalent:
1. The identity $s = t$ holds in every aperiodic finite monoid and
2. $[s]_\pi = [t]_\pi$.
\end{prop}

\begin{proof}
We show both implications separately.

"1. $\Rightarrow$ 2."
The results in [9] imply that the identity $s = t$ can be deduced from the following list of axioms, where $n \geq 1$:

\begin{align*}
(uv)w &= u(vw) \\
\pi^\pi &= u^\pi \\
\omega^\pi &= \omega^\pi \\
\omega^\omega^\omega^\omega^\omega &= u(\nu\nu) \\
\eta^\omega &= \omega^\omega^\omega^\omega^\omega \\
\eta^\omega^\omega^\omega^\omega^\omega &= u(\nu\nu) \\
\eta^\omega^\omega^\omega^\omega^\omega &= u(\nu\nu)
\end{align*}

As a matter of fact, the $\pi$-power algebra on $\Lambda^{LO}$ satisfies these axioms as well. Consequently, $[s]_\pi = [t]_\pi$ can be proved along a deduction of $s = t$ from the axioms.

"2. $\Rightarrow$ 1."
Due to Eilenberg’s Variety Theorem [5], the pseudovariety of aperiodic monoids is generated by the class of syntactic aperiodic monoids. The latter are precisely the syntactic monoids of first-order definable languages [10, 14]. Thus, it suffices to show that $s = t$ holds in the syntactic monoid of every such language. This follows directly from Theorem 4.1.

\begin{proofofthm}
In order to apply the decision procedure from [3, Theorem 79], we have to translate $s$ and $t$ into expressions generating the same words and which do not use $\pi$-power but $\omega$-power, $\omega^*$-power and dense shuffle instead. Such a translation can be based on the identity $u^\pi = u^\omega[u^\omega^*u^\omega^*]_{\eta} u^\omega^*$ which holds for all words $u \in \Lambda^{LO}$. Therein, $\eta$-power is a special case of the dense shuffle. Since this translation leads to a blow-up which is exponential in the number of nested applications of $\pi$-power in $s$ and $t$, we can decide $[s]_\pi = [t]_\pi$ in time at most exponential in the size of $s$ and $t$.
\end{proofofthm}
Summary

For every $\pi$-term $t$ we define a labeled linear order $[t]_\pi$, and every first-order fragment $F$ over finite words naturally yields a (possibly infinite) Ehrenfeucht-Fraïssé game on labeled linear orders. We note that $[t]_\pi$ does not depend on $F$. The important property of these constructions is that $F$ satisfies an identity $s = t$ of $\pi$-terms $s$ and $t$ if and only if duplicator has a winning strategy in the $F$-game on $[s]_\pi$ and $[t]_\pi$. Usually showing that a fragment $F$ satisfies an identity $s = t$ requires a significant amount of book-keeping which in most cases is not part of the actual proof idea. Our main results Theorem 4.1 and Theorem 4.2 allow to formalize such proof ideas without further book-keeping, see e.g. Corollary 4.7. A probably less obvious application of our main result are word problems for $\pi$-terms over varieties of finite monoids. We show that the word problem for $\pi$-terms over aperiodic finite monoids is solvable in exponential time (Theorem 5.1), thereby improving a result of McCammond [9].

Several possible extensions of our result come to mind: Other implicit operations (see [1] for further details on implicit operations), logical fragments beyond classical first-order logic, and other structures such as infinite words, trees or Mazurkiewicz traces.

References

A Missing Proofs from Section 3

A.1 Missing Proofs for Theorem 3.1

Theorem 3.1. Let \( \mathcal{F} \) be a fragment of bounded quantifier rank. For all words \( u, v \in \Lambda^\omega \) the following are equivalent:
1. \( u \models \varphi \) implies \( v \models \varphi \) for all sentences \( \varphi \in \mathcal{F} \) and
2. duplicator has a winning strategy in the \( \mathcal{F} \)-game on \( (u, v) \).

As indicated earlier, the implication “2. \( \Rightarrow \) 1.” even holds if the quantifier rank of \( \mathcal{F} \) is not bounded. The contraposition of this claim is shown by the proposition below.

Proposition A.1. Let \( \mathcal{F} \) be a fragment and \( (u, \alpha), (v, \beta) \) \( \mathcal{X} \)-valuations. If there exists a formula \( \varphi \in \mathcal{F} \) with \( \text{IV}(\varphi) \subseteq \mathcal{X} \) such that \( (u, \alpha) \models \varphi \) but \( (v, \beta) \not\models \varphi \), then \( (u, \alpha) \not\leq_{\mathcal{F}} (v, \beta) \).

Proof. Using De Morgan’s laws, we can transform \( \varphi \) into a positive Boolean combination of literals and formulae starting with one of the quantifiers in \( Q \). Due to the syntactic closure properties of fragments, the resulting formula also belongs to \( \mathcal{F} \). Thus, we may assume without loss of generality that \( \varphi \) itself is a literal or a quantified formula. We proceed by induction on \( \text{qr}(\varphi) \).

Base case: \( \text{qr}(\varphi) = 0 \), i.e., \( \varphi \) is a literal. The assumption that \( (u, \alpha) \models \varphi \) but \( (v, \beta) \not\models \varphi \) means that the situation \( (\mathcal{F}, (u, \alpha), (v, \beta)) \) satisfies spoilers winning condition.

Inductive step: \( \text{qr}(\varphi) > 0 \), i.e., \( \varphi \) is a quantified formula. Let \( \varphi = Qx \psi \) with \( Q \in Q \), \( x \in \mathcal{V} \) and \( \psi \in \mathcal{F}/Qx \). We only demonstrate the case \( Q = \exists \). Since \( (u, \alpha) \models \varphi \), there exists a \( q \in \text{dom}(u) \) such that \( (u, \alpha[x/q]) \models \psi \). Let spoiler choose quantifier \( \exists x \) and quest \( q \) in situation \( (\mathcal{F}, (u, \alpha), (v, \beta)) \). If \( \text{dom}(v) = \emptyset \), he immediately wins the game. Otherwise, suppose that duplicator replies by \( r \in \text{dom}(v) \). Since \( (v, \beta) \not\models \varphi \), we obtain \( (v, \beta[x/r]) \not\models \psi \). By the induction hypothesis, this implies \( (u, \alpha[x/q]) \not\leq_{\mathcal{F}/\exists x} (v, \beta[x/r]) \) and in turn \( (u, \alpha) \not\leq_{\mathcal{F}} (v, \beta) \).

The following lemma is a well-known fact about first-order logic.

Lemma A.2. Let \( n \in \mathbb{N} \) and \( \mathcal{X} \subseteq \mathcal{V} \) be a finite set of variables. There exists a finite subset \( \Phi \subseteq \text{FO}_n \) of formulae \( \varphi \) with \( \text{IV}(\varphi) \subseteq \mathcal{X} \) such that every formula \( \psi \in \text{FO}_n \) with \( \text{IV}(\psi) \subseteq \mathcal{X} \) is equivalent to a formula in \( \Phi \).

Corollary A.3. Let \( \mathcal{F} \) be a fragment of bounded quantifier rank and \( \mathcal{X} \subseteq \mathcal{V} \) a finite set of variables. There exists a finite subset \( \Phi \subseteq \mathcal{F} \) of formulae \( \varphi \) with \( \text{IV}(\varphi) \subseteq \mathcal{X} \) such that every formula \( \psi \in \mathcal{F} \) with \( \text{IV}(\psi) \subseteq \mathcal{X} \) is equivalent to a formula in \( \Phi \).

Proof. Let \( n \in \mathbb{N} \) be a bound on the quantifier rank of \( \mathcal{F} \) and \( \Psi \subseteq \text{FO}_n \) one of the finite sets whose existence is guaranteed by Lemma A.2. Let \( \Xi \subseteq \Psi \) be the set of those \( \xi \in \Psi \) for which there exists an equivalent formula \( \varphi_{\xi} \in \mathcal{F} \) with \( \text{IV}(\varphi_{\xi}) \subseteq \mathcal{X} \). We show the finite set \( \Phi = \{ \varphi_{\xi} \mid \xi \in \Xi \} \) to have the desired property. Consider an arbitrary formula \( \psi \in \mathcal{F} \) with \( \text{IV}(\psi) \subseteq \mathcal{X} \). Due to the choice of \( \Psi \), there exists a formula \( \xi \in \Psi \) which is equivalent to \( \psi \). In particular, \( \xi \in \Xi \). Thus, the formula \( \varphi_{\xi} \in \Phi \) is equivalent to \( \psi \) as well.

Proposition A.4. Let \( \mathcal{F} \) be a fragment of bounded quantifier rank and \( (u, \alpha), (v, \beta) \) \( \mathcal{X} \)-valuations. If \( (u, \alpha) \models \varphi \) implies \( (v, \beta) \models \varphi \) for all formulae \( \varphi \in \mathcal{F} \) with \( \text{IV}(\varphi) \subseteq \mathcal{X} \), then \( (u, \alpha) \leq_{\mathcal{F}} (v, \beta) \).
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Proof. Let $n \in \mathbb{N}$ be a bound on the quantifier rank of $F$. We proceed by induction on $n$.

Base case: $n = 0$. By assumption, the situation $(F, \langle u, \alpha \rangle, \langle v, \beta \rangle)$ does not satisfy spoiler’s winning condition. Since $F$ contains no quantified formulae, this implies $\langle u, \alpha \rangle \preceq_F \langle v, \beta \rangle$.

Inductive step: $n > 0$. Suppose that spoiler chooses quantifier $Qx$ and quest $q$ in situation $(F, \langle u, \alpha \rangle, \langle w, \gamma \rangle)$. We are looking for a suitable reply $r$ for duplicator. We only demonstrate the case $Q = \exists$, where $q \in \text{dom}(u)$. By Corollary A.3, there exists a finite set $\Phi \subseteq F/\exists x$ of formulae $\varphi$ with $\text{FV}(\varphi) \subseteq \mathbb{N} \cup \{x\}$ such that every formula $\psi \in F/\exists x$ with $\text{FV}(\psi) \subseteq \mathbb{N} \cup \{x\}$ is equivalent to a formula in $\Phi$. We consider the formulae

$$
\xi = \bigwedge_{\varphi \in \Phi, \langle u, \alpha[x/q]\rangle \models \varphi} \varphi \quad \text{and} \quad \psi = \exists x \xi.
$$

Notice that $\xi \in F/\exists x$ and $\chi \in F$. Due to the choice of $\xi$ and $\chi$, we have $\langle u, \alpha[x/q]\rangle \models \xi$ and hence $\langle u, \alpha \rangle \models \chi$. Using the assumption of the proposition, we conclude $\langle v, \beta \rangle \models \chi$. Thus, there is an $r \in \text{dom}(v)$ such that $\langle v, \beta[x/r]\rangle \models \xi$. We show that $r$ is a suitable reply for duplicator, i.e., $r$ satisfies $\langle u, \alpha[x/q]\rangle \preceq_{F/\exists x} \langle v, \beta[x/r]\rangle$. By the induction hypothesis and due to the choice of $\Phi$, it suffices to show that all formulae $\varphi \in \Phi$ with $\langle u, \alpha[x/q]\rangle \models \varphi$ also satisfy $\langle v, \beta[x/r]\rangle \models \varphi$. In fact, this follows directly from the choice of $\xi$ and $\langle v, \beta[x/r]\rangle \models \xi$. ◀

A.2 Missing Proofs for Theorem 3.3

▶ Theorem 3.3. Let $F$ be a suc-stable fragment. For all $\omega$-rational words $u, v \in \Lambda^\omega_0$ the following are equivalent:

1. $u \models \varphi$ implies $v \models \varphi$ for all sentences $\varphi \in F$ and
2. duplicator has a winning strategy in the $F$-game on $(u, v)$.

In the course of demonstrating this theorem, we have to explicitly construct winning strategies for duplicator several times. The general pattern behind the constructions always consists of the following four steps (cf. the proof of Theorem 3.3):

1. We declare a class of situations of the game which we call good for duplicator.
2. We show that the starting situation of the game is good.
3. We show that no good situation satisfies the winning condition of spoiler.
4. We show that for every good situation and any choice of a quantifier and a quest by spoiler, there exists a reply by duplicator such that the resulting situation is again good. Whenever these four steps can be implemented, duplicators winning strategy is to simply stay within the class of good situations.

▶ Proposition A.5. Let $F$ be a fragment. The relation $\preceq_F$ is transitive.

Proof. Consider $\mathbb{N}$-valuations $\langle u, \alpha \rangle, \langle v, \beta \rangle, \langle w, \gamma \rangle$ with $\langle u, \alpha \rangle \preceq_F \langle v, \beta \rangle$ and $\langle v, \beta \rangle \preceq_F \langle w, \gamma \rangle$. We have to show $\langle u, \alpha \rangle \preceq_F \langle w, \gamma \rangle$. For this purpose, we adhere closely to the four steps of constructing a winning strategy for duplicator.

1. We call a $\mathbb{N}$-situation $(G, \langle u, \alpha' \rangle, \langle w, \gamma' \rangle)$ good if there exists a $\mathbb{N}$-valuation $\langle v, \beta' \rangle$ such that $\langle u, \alpha' \rangle \preceq_G \langle v, \beta' \rangle$ and $\langle v, \beta' \rangle \preceq_G \langle w, \gamma' \rangle$.
2. Clearly, the situation $(F, \langle u, \alpha, \rangle \langle w, \gamma \rangle)$ is good.
3. Let $S = (G, \langle u, \alpha', \rangle \langle w, \gamma' \rangle)$ be a good $\mathbb{N}$-situation and $\langle v, \beta' \rangle$ a witnessing $\mathbb{N}$-valuation. For every literal $\varphi \in G$ with $\text{FV}(\varphi) \subseteq \mathbb{N}$ and $\langle u, \alpha' \rangle \models \varphi$, the choice of $\beta'$ implies $\langle v, \beta' \rangle \models \varphi$ and in turn $\langle w, \gamma' \rangle \models \varphi$. Thus, $S$ does not satisfy spoiler’s winning condition.
4. Suppose that spoiler chooses quantifier $Qx$ and quest $q$ in a good situation $S = (G, \langle u, \alpha' \rangle, \langle w, \gamma' \rangle)$. We are looking for a suitable reply $r$ for duplicator. We only demonstrate the case $Q = \exists$, where $q \in \text{dom}(u)$. Let $\langle v, \beta' \rangle$ be a valuation witnessing that $S$ is good. Since $\langle u, \alpha' \rangle \preceq_G \langle v, \beta' \rangle$ and $\langle v, \beta' \rangle \preceq_G \langle w, \gamma' \rangle$, there exist a $p \in \text{dom}(v)$ such that $\langle u, \alpha'[x/q] \rangle \preceq_{G[\exists x]} \langle v, \beta'[x/p] \rangle$ and an $r \in \text{dom}(w)$ such that $\langle v, \beta'[x/p] \rangle \preceq_{G[\exists x]} \langle w, \gamma'[x/r] \rangle$. Thus, $S[\exists x, q, r]$ is a good situation. 

The following two lemmas are auxiliary statements for the proof of Proposition 3.4.

**Lemma A.6.** Let $F$ be a fragment, $\langle u, \alpha \rangle, \langle v, \beta \rangle$ $X$-valuations and $\langle u, \alpha' \rangle, \langle v, \beta' \rangle$ their respective restrictions to $X'$-valuations for some subset $X' \subseteq X$. If $\langle u, \alpha \rangle \preceq_F \langle v, \beta \rangle$, then $\langle u, \alpha' \rangle \preceq_F \langle v, \beta' \rangle$.

**Proof.** We adhere closely to the four steps of constructing a winning strategy for duplicator.

1. A $\forall'$-situation $(G, \langle u, \gamma' \rangle, \langle v, \delta' \rangle)$ is good if there exist a finite set $Y \subseteq V$ with $Y \supseteq \forall'$ and $\forall'$-valuations $\langle u, \gamma \rangle, \langle v, \delta \rangle$ extending $\langle u, \gamma' \rangle, \langle v, \delta' \rangle$ such that $\langle u, \gamma \rangle \preceq_F \langle v, \delta \rangle$.

2. Clearly, the situation $(\mathcal{F}, \langle u, \alpha' \rangle, \langle v, \beta' \rangle)$ is good.

3. Let $S = (G, \langle u, \gamma' \rangle, \langle v, \delta' \rangle)$ be a $\forall'$-situation and $\langle u, \gamma \rangle, \langle v, \delta \rangle$ witnessing $\forall'$-valuations. Consider a literal $\varphi \in G$ with $\text{IV}(\varphi) \subseteq \forall'$ and $\langle u, \gamma' \rangle \models \varphi$. We have to show $\langle v, \delta' \rangle \models \varphi$. Since $\gamma'$ is a restriction of $\gamma$, we obtain $\langle u, \gamma \rangle \models \varphi$. The assumption $\langle u, \gamma \rangle \preceq_G \langle v, \delta \rangle$ implies $\langle v, \delta \rangle \models \varphi$. As $\delta'$ is a restriction of $\delta$, we conclude $\langle v, \delta' \rangle \models \varphi$. Thus, $S$ does not satisfy spoiler’s winning condition.

4. Suppose that spoiler chooses quantifier $Qx$ and quest $q$ in a good situation $S = (G, \langle u, \gamma' \rangle, \langle v, \delta' \rangle)$. We are looking for a suitable reply $r$ for duplicator. We only demonstrate the case $Q = \exists$, where $q \in \text{dom}(u)$. Let $\langle u, \gamma \rangle, \langle v, \delta \rangle$ be situations witnessing that $S$ is good. Due to $\langle u, \gamma \rangle \preceq_G \langle v, \delta \rangle$, there is an $r \in \text{dom}(v)$ such that $\langle u, \gamma[x/q] \rangle \preceq_{G[\exists x]} \langle v, \delta[x/r] \rangle$. Since $\langle u, \gamma'[x/q] \rangle, \langle v, \delta'[x/r] \rangle$ are restrictions of $\langle u, \gamma[x/q] \rangle, \langle v, \delta[x/r] \rangle$, the situation $S[\exists x, q, r]$ is good.

**Lemma A.7.** Let $F$ be a fragment and $\langle u, \alpha \rangle, \langle v, \beta \rangle$ $X$-valuations with $\langle u, \alpha \rangle \preceq_F \langle v, \beta \rangle$. Let $Q \in Q$ and $x \in V \setminus X$ such that $F/Qx \neq \emptyset$.

1. If $Q \in \{\exists, \forall\}$ is a non-negated quantifier, then $\langle u, \alpha \rangle \preceq_{F/Qx} \langle v, \beta \rangle$.

2. If $Q \in \{\neg \exists, \neg \forall\}$ is a negated quantifier, then $\langle v, \beta \rangle \preceq_{F/Qx} \langle u, \alpha \rangle$.

**Proof.** We only demonstrate the case $Q = \exists$. First, suppose that $\text{dom}(u) \neq \emptyset$. Let $q \in \text{dom}(u)$ be arbitrary. Due to the assumption $\langle u, \alpha \rangle \preceq_F \langle v, \beta \rangle$, there exists an $r \in \text{dom}(v)$ such that $\langle u, \alpha[x/q] \rangle \preceq_{F[\exists x]} \langle v, \beta[x/r] \rangle$. Since $\alpha$ and $\beta$ are the restrictions of $\alpha[x/q]$ and $\beta[x/r]$ to $X$, respectively, Lemma A.6 yields the claim.

Now, assume that $\text{dom}(u) = \emptyset$. If $\text{dom}(v) = \emptyset$ as well, then $\langle u, \alpha \rangle = \langle v, \beta \rangle$ and the claim is trivial. Thus, we further assume $\text{dom}(v) \neq \emptyset$. Let $\varphi \in F[\exists x]$ be a literal with $\text{IV}(\varphi) \subseteq X$ and $\langle u, \alpha \rangle \models \varphi$. Since $x \notin \text{IV}(\varphi)$, we have $\varphi \in F$. Thus, $\langle u, \alpha \rangle \preceq_F \langle v, \beta \rangle$ implies $\langle v, \beta \rangle \models \varphi$. Thus, $S = (F[\exists x], \langle u, \alpha \rangle, \langle v, \beta \rangle)$ does not satisfy spoiler’s winning condition. We show that he even cannot choose both a quantifier $Qy$ and a quest $q$ in situation $S$. If spoiler chooses $Q \in \{\exists, \neg \forall\}$, he cannot choose a quest $q$ and immediately loses. However, if he could choose $Q \in \{\forall, \neg \exists\}$, we had $Qy: T \in F[\exists x]$ and hence $Qy: T \in F$ because $x \notin \text{IV}(Qy: T)$. But this would contradict $\langle u, \alpha \rangle \preceq_F \langle v, \beta \rangle$. Thus, spoiler cannot do any move in situation $S$ and duplicator immediately wins, i.e., $\langle u, \alpha \rangle \preceq_{F[\exists x]} \langle v, \beta \rangle$.

**Proposition 3.4.** Let $F$ be a suc-stable fragment, $k \in \mathbb{N}$ and $\langle u_i, \alpha_i \rangle, \langle v_i, \beta_i \rangle$ $X_i$-valuations with mutually disjoint $X_i$ for $i \in [1, k]$. If $\langle u_i, \alpha_i \rangle \preceq_F \langle v_i, \beta_i \rangle$ for each $i \in [1, k]$, then $\langle u_1 \ldots u_k, \alpha_1 \cup \ldots \cup \alpha_k \rangle \preceq_F \langle v_1 \ldots v_k, \beta_1 \cup \ldots \cup \beta_k \rangle$. 


Proof. Let \( u = u_1 \ldots u_k \), \( \alpha = \alpha_1 \cup \cdots \cup \alpha_k \), \( v = v_1 \cdots v_k \), and \( \beta = \beta_1 \cup \cdots \cup \beta_k \). We have to show \( \langle u, \alpha \rangle \preceq_F \langle v, \beta \rangle \). For this purpose, we adhere closely to the four steps of constructing a winning strategy for duplicator as described above.

1. A \( \forall \)-situation \((G, \langle u, \alpha \rangle, \langle v, \beta \rangle)\) is good if there exist a partition \( \forall = \forall_1 \sqcup \cdots \sqcup \forall_k \) and \( \forall_i \)-valuations \( \langle u_i, \alpha_i \rangle, \langle v_i, \beta_i \rangle \) for \( i \in [1, k] \) such that \( \alpha' = \alpha'_1 \cup \cdots \cup \alpha'_k \) and \( \beta' = \beta'_1 \cup \cdots \cup \beta'_k \) as well as \( \langle u_i, \alpha'_i \rangle \preceq_G \langle v_i, \beta'_i \rangle \) for each \( i \in [1, k] \).

2. Clearly, the situation \((F, \langle u, \alpha \rangle, \langle v, \beta \rangle)\) is good.

3. Let \((G, \langle u, \alpha \rangle, \langle v, \beta \rangle)\) be a good \( \forall \)-situation and \( \langle u_i, \alpha'_i \rangle, \langle v_i, \beta'_i \rangle \) witnessing \( \forall_i \)-valuations for \( i \in [1, k] \). We have to show that all literals \( \varphi \in G \) with \( Fv(\varphi) \subseteq \forall \) and \( \langle u, \alpha \rangle \models \varphi \) also satisfy \( \langle v, \beta \rangle \models \varphi \). The only literals for which this is a non-trivial task are \( \text{succ}(x, y) \), \( \text{min}(x) \) and \( \text{max}(x) \) as well as their negations. We only demonstrate the implication for \( \text{succ}(x) \). The cases \( \text{min}(x) \) and \( \text{max}(x) \) as well as the negative literals are treated similarly.

Consider the unique \( i, j \in [1, k] \) with \( x \in \forall_i \) and \( y \in \forall_j \). Since \( \langle u, \alpha \rangle \models \text{succ}(x, y) \), the case \( i > j \) cannot occur. Henceforth, we assume \( i < j \). We first conclude \( \langle u_i, \alpha'_i \rangle \models \text{max}(x) \), \( \langle u_i, \alpha'_i \rangle \models \text{empty} \) for all \( \ell \in [i + 1, j - 1] \), and \( \langle u_i, \alpha'_i \rangle \models \text{min}(y) \). The suc-stability of \( G \) implies \( \text{max}(x), \text{min}(y), \emptyset \in G \). Since \( \langle u_i, \alpha'_i \rangle \preceq_G \langle v_i, \beta'_i \rangle \) for all \( \ell \in [1, k] \), we obtain \( \langle v_i, \beta'_i \rangle \models \text{max}(x) \), \( \langle v_i, \beta'_i \rangle \models \text{empty} \) for all \( \ell \in [i + 1, j - 1] \), and \( \langle v_i, \beta'_i \rangle \models \text{min}(y) \). Finally, we conclude \( \langle v, \beta \rangle \models \text{succ}(x, y) \).

4. Suppose that spoiler chooses quantifier \( Q_x \) and quest \( q \) in a good situation \( S = (G, \langle u, \alpha \rangle, \langle v, \beta \rangle) \). We are looking for a suitable reply \( r \) for duplicator. We only demonstrate the case \( Q = \exists \), where \( q \in \text{dom}(u) \). Let \( \langle u_i, \alpha'_i \rangle, \langle v_i, \beta'_i \rangle \) for \( i \in [1, k] \) be valu-ations witnessing that \( S \) is good. There is a unique \( i \in [1, k] \) such that \( q \in \text{dom}(u_i) \). Let \( r \in \text{dom}(v_i) \) be such that \( \langle u_i, \alpha'_i[x/q] \rangle \preceq_{G/\exists x} \langle v_i, \beta'_i[x/r] \rangle \). Due to Lemmas A.6 and A.7, we further have \( \langle u_i, \alpha'_i[x/q] \rangle \preceq_{G/\exists x} \langle v_i, \beta'_i[x/r] \rangle \) for all \( \ell \in [1, k] \) with \( \ell \neq i \), where \( \alpha''_i \) and \( \beta''_i \) are the restrictions of \( \alpha'_i \) and \( \beta'_i \) to \( \forall \setminus \{x\} \), respectively. Putting \( \alpha''_i = \alpha'_i[x/q] \) and \( \beta''_i = \beta'_i[x/r] \), we obtain

\[
\alpha'[x/q] = \alpha''_1 \cup \cdots \cup \alpha''_k \quad \text{and} \quad \beta'[x/q] = \beta''_1 \cup \cdots \cup \beta''_k.
\]

Thus, \( S[\exists x, q, r] \) is a good situation and hence \( r \) is a suitable reply for duplicator.

Lemma 3.5. Let \( F \) be a \text{su}c-stable fragment whose quantifier rank is bounded by \( n \in \mathbb{N} \) and \( u, v \in A^{10} \). If \( u \preceq_F v \), then \( u^m \preceq_F v^m \) and \( u^\varpi \preceq_F v^\varpi \) for all \( m \geq 2^n - 1 \) but \( m = 0 \).

Proof. We only show the claim \( u^m \preceq_F v^m \). The proof of the other claim is similar. We proceed by induction on \( n \).

Base case: \( n = 0 \). There are at most six literals \( \varphi \in F \) without free variables, namely \( \top, \bot \) and empty as well as their negations. For all of them, one easily shows that \( u^m \models \varphi \) implies \( v^m \models \varphi \). Thus, the situation \((F, u^m, v^m)\) does not satisfy spoiler’s winning condition. Since \( F \) contains no quantified formulae, duplicator immediately wins in this situation.

Inductive step: \( n > 0 \). We assume that \( \text{dom}(u^m) = \text{dom}(u) \times [1, m] \). Let \( (P, \preceq) \) be a linear ordering of isomorphism type \( \varpi \) such that \( \text{dom}(u) = \text{dom}(v) \times P \). For every \( p \in P \) let \( \varpi_p \) and \( \overline{\varpi}_p \) be the order types of the suborders of \( (P, \preceq) \) induced by the open intervals \((\!-\!\infty, p)\) and \((p, +\infty)\), respectively. Then \( \varpi = \varpi_p + 1 + \overline{\varpi}_p \).

Suppose that spoiler chooses quantifier \( Q_x \) and quest \( q \) in situation \((F, u^m, v^m)\). We are looking for a suitable reply \( r \) for duplicator. We only demonstrate the cases \( Q = \exists \) and \( Q = \forall \). Recall that Lemma A.7 implies \( u \preceq_{F/Q_x} v \) in these two cases.
Inductive step 1: Proof. We proceed by induction on the structure of $\varphi$. Since $\langle u, [x/s] \rangle \preceq_{\mathcal{F}/3n} \langle v, [x/t] \rangle$. Depending on $k$ we will choose a $p \in P$ such that

$$u^{k-1} \preceq_{\mathcal{F}/3x} u^p \quad \text{and} \quad u^{m-k} \preceq_{\mathcal{F}/3x} v^p.$$  

(1)

By Proposition 3.4, this will imply $\langle u^m, [x/(s,k)] \rangle \preceq_{\mathcal{F}/3x} \langle v^m, [x/(t,p)] \rangle$, i.e., $r = (t,p)$ is a suitable reply for duplicator.

Case 1: $Q = \exists$, where $q = (s,k) \in \text{dom}(u)$. Since $u \preceq_{\mathcal{F}/Qx} v$, There exists a $t \in \text{dom}(v)$ such that $\langle u, [x/s] \rangle \preceq_{\mathcal{F}/3n} \langle v, [x/t] \rangle$. Depending on $k$ we will choose a $p \in P$ such that

$$u^{k-1} \preceq_{\mathcal{F}/3x} u^p \quad \text{and} \quad u^{m-k} \preceq_{\mathcal{F}/3x} v^p.$$  

(2)

Again by Proposition 3.4, this will imply $\langle u^m, [x/(s,k)] \rangle \preceq_{\mathcal{F}/3x} \langle v^m, [x/(t,p)] \rangle$, i.e., $r = (s,k)$ will be a suitable reply for duplicator.

Case 1a: $k \leq m - (2^{n-1} - 1)$. Observe that $m - k \geq 2^{n-1} - 1$. We choose $p \in P$ such that $\overline{r}_p = k - 1$ and $\overline{r}_p = \varpi$. The conditions in Eq. (1) are met due to Proposition 3.4 and the induction hypothesis, respectively.

Case 1b: $k > m - (2^{n-1} - 1)$. Observe that $k - 1 > 2^{n-1} - 1$. We choose $p \in P$ such that $\overline{r}_p = \varpi$ and $\overline{r}_p = m - k$. The conditions in Eq. (1) are met due to the induction hypothesis and Proposition 3.4, respectively.

Case 1d: $\sigma = \varpi$ and $r_p$ is finite. This case is symmetric to Cases 2a and 2b.

Case 2: $Q = \forall$, where $q = (t,p) \in \text{dom}(v)$. We proceed similarly to Case 1. There exists an $s \in \text{dom}(u)$ such that $\langle u, [x/s] \rangle \preceq_{\mathcal{F}/3x} \langle v, [x/t] \rangle$. Depending on $p$ we will choose a $k \in [1,m]$ such that

$$u^{k-1} \preceq_{\mathcal{F}/3x} v^p \quad \text{and} \quad u^{m-k} \preceq_{\mathcal{F}/3x} v^p.$$  

(2)

Again by Proposition 3.4, this will imply $\langle u^m, [x/(s,k)] \rangle \preceq_{\mathcal{F}/3x} \langle v^m, [x/(t,p)] \rangle$, i.e., $r = (s,k)$ will be a suitable reply for duplicator.

Case 2a: $\overline{r}_p$ is finite with $\overline{r}_p < 2^{n-1} - 1$ and $\overline{r}_p = \varpi$. We choose $k = \overline{r}_p + 1 \leq 2^{n-1} - 1$. Thus, $m - k \geq 2^{n-1}$. The conditions in Eq. (1) are met due to Proposition 3.4 and the induction hypothesis.

Case 2b: $\overline{r}_p$ is finite with $\overline{r}_p \geq 2^{n-1} - 1$ and $\overline{r}_p = \varpi$. We choose $k = 2^{n-1}$. Consequently, $m - k \geq 2^{n-1} - 1$. The induction hypothesis implies $u^{k-1} \preceq_{\mathcal{F}/3x} u^p$ and $u^m \preceq_{\mathcal{F}/3x} v^p$. The conditions in Eq. (1) are met by transitivity and the induction hypothesis.

Case 2c: $\overline{r}_p = \overline{r}_p = \varpi$. We choose $k = 2^{n-1}$. Then $m - k \geq 2^{n-1} - 1$. The conditions in Eq. (1) both are met due to the induction hypothesis.

B Missing Proofs from Section 4

Corollary 4.5. Let $\mathcal{F}$ be a suc-stable fragment whose quantifier rank is bounded by $n \in \mathbb{N}$ and let $t \in T_\Lambda$ be a $\pi$-term. Then $\llbracket t \rrbracket^a = \llbracket t \rrbracket^a$ for all $m \geq 2^n - 1$ but $m = 0$.

Proof. We proceed by induction on the structure of $t$.

Base case: $t = a$ for some $a \in A$. Since $\llbracket t \rrbracket^a = \llbracket t \rrbracket^a$ is trivial.

Inductive step 1: $t = s_1s_2$ for $\pi$-terms $s_1, s_2 \in T_\Lambda$. Due to the induction hypothesis, we have $\llbracket s_i \rrbracket^a \approx_{\mathcal{F}} \llbracket s_i \rrbracket^a$ for $i = 1, 2$. Two applications of Proposition 3.4 yield $\llbracket t \rrbracket^a = \llbracket s_1 \rrbracket^a \llbracket s_2 \rrbracket^a = \llbracket t \rrbracket^a$.

Inductive step 2: $t = s^a$ for a $\pi$-term $s \in T_\Lambda$. Due to the induction hypothesis, we have $\llbracket s \rrbracket^a \approx_{\mathcal{F}} \llbracket s \rrbracket^a$. Applying Lemma 3.5 twice yields $\llbracket t \rrbracket^a = \llbracket s \rrbracket^a \llbracket s \rrbracket^a = \llbracket t \rrbracket^a$. \hfill \blacksquare
C Missing Proofs from Section 5

The proof of Proposition 5.2 lacks evidence of the following claim.

▶ Lemma C.1. The following identities hold for all \( n \geq 1 \) and \( u, v, w \in \Lambda^{LO} \):

\[
\begin{align*}
(uv)w &= u(vw) & \text{(Ax1)} \\
(u^n)v &= u^nv & \text{(Ax2)} \\
(u^n)^v &= u^v & \text{(Ax3)} \\
w^vu^v &= u^v & \text{(Ax4)} \\
w^vu = u^vw = u^v & \text{(Ax5)} \\
(uv)^v = u(vu)^v & \text{(Ax6)}
\end{align*}
\]

Proof. It is a matter of routine to check that the concatenation of words is associative, i.e., (Ax1) holds. It is equally simple to check that the following two laws of exponentiation hold for any word \( w \in \Lambda^{LO} \) and all order types \( \sigma \) and \( \tau \):

\[
\begin{align*}
w^{\sigma+\tau} &= w^{\sigma}w^{\tau} & \text{and} & & w^{\sigma\cdot\tau} &= (w^{\sigma})^{\tau}.
\end{align*}
\]

To (Ax2). The three identities below hold for all order types \( \sigma \) and \( \tau \):

\[
\begin{align*}
(\sigma + \tau) \cdot \omega &= \sigma + (\tau + \sigma) \cdot \omega, \\
(\sigma + \tau) \cdot \zeta &= (\tau + \sigma) \cdot \zeta, \\
(\sigma + \tau) \cdot \omega^* &= (\tau + \sigma) \cdot \omega^* + \tau.
\end{align*}
\]

We conclude

\[
\begin{align*}
\omega \cdot \omega &= (\omega + \zeta \cdot \eta + \omega^*) \cdot (\omega + \zeta \cdot \eta + \omega^*) \\
&= (\omega + \zeta \cdot \eta + \omega^*) \cdot \omega + (\omega + \zeta \cdot \eta + \omega^*) \cdot \zeta \cdot \eta + (\omega + \zeta \cdot \eta + \omega^*) \cdot (\omega + \zeta \cdot \eta + \omega^*) \\
&= \omega + \zeta \cdot (\eta + 1) \cdot (\omega + \zeta \cdot (\eta + 1) \cdot \zeta \cdot \eta + \zeta \cdot (1 + \eta) \cdot \omega^* + \omega^*) \\
&= \omega + \zeta \cdot \eta + \zeta \cdot \eta + \zeta \cdot \eta + \omega^* \\
&= \omega + \zeta \cdot (\eta + \eta \cdot \eta + \eta) + \omega^* \\
&= \omega.
\end{align*}
\]

Thus,

\[
(u^n)^v = u^v u^v = u^v.
\]

To (Ax3). We use the following tree identities:

\[
\begin{align*}
n \cdot \omega &= \omega, & n \cdot \zeta &= \zeta, & n \cdot \omega^* &= \omega^*.
\end{align*}
\]

Consequently,

\[
n \cdot \omega = n \cdot (\omega + \zeta \cdot \eta + \omega^*) = n \cdot \omega + n \cdot \zeta \cdot \eta + n \cdot \omega^* = \omega + \zeta \cdot \eta + \omega^* = \omega,
\]

and hence

\[
(u^n)^v = u^n u^v = u^v.
\]
To (Ax4). We have
\[\varepsilon + \varepsilon = \omega + \zeta \cdot \eta + \omega^* + \omega + \zeta \cdot \eta + \omega^* = \omega + \zeta \cdot (\eta + 1 + \eta) + \omega^* = \varepsilon\]
and therefore
\[u^\varepsilon u^\varepsilon = u^{\varepsilon + \varepsilon} = u^\varepsilon.\]

To (Ax5). We have
\[\varepsilon + 1 = \omega + \zeta \cdot \eta + \omega^* + 1 = \varepsilon\quad \text{and} \quad 1 + \varepsilon = 1 + \omega + \zeta \cdot \eta + \omega^* = \varepsilon.\]
Consequently,
\[u^\varepsilon u = u^{\varepsilon + 1} = u^\varepsilon\quad \text{and} \quad uu^\varepsilon = u^{1 + \varepsilon} = u^\varepsilon.\]

To (Ax6). For all \(a, b \in \Lambda\), the following identities are easy to check:
\[(ab)^\omega = a(ba)^\omega, \quad (ab)^\zeta = (ba)^\zeta, \quad (ba)^{\omega^*} = (ab)^{\omega^*} a.\]
Replacing in the words occurring in these identities every \(a\)-labelled position by the word \(u\) and every \(b\)-labelled position by the word \(v\), yields the following identities:
\[(uv)^\omega = u(vu)^\omega, \quad (uv)^\zeta = (vu)^\zeta, \quad (vu)^{\omega^*} = (uv)^{\omega^*} u.\]
Thus,
\[(uv)^\varepsilon u = (uv)^\varepsilon ((uv)^\zeta)^\eta (uv)^{\omega^*} u = u(vu)^\varepsilon ((vu)^\zeta)^\eta (vu)^{\omega^*} = u(vu)^\varepsilon. \quad \blacktriangleleft\]