A KLEENE-SCHÜTZENBERGER THEOREM FOR TRACE SERIES OVER BOUNDED LATTICES

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Abstract
We study weighted trace automata with weights in strong bimonoids. Traces form a generalization of words that allow to model concurrency; strong bimonoids are algebraic structures that can be regarded as “semirings without distributivity”. A very important example for the latter are bounded lattices, especially non-distributive ones. We show that if both operations of the bimonoid are locally finite, then the classes of recognizable and mc-rational trace series coincide and, in general, are properly contained in the class of c-rational series. Moreover, if, in addition, in the bimonoid the addition is idempotent and the multiplication is commutative, then all three classes coincide.

1. Introduction

In the theory of automata and formal languages, Kleene’s foundational theorem on the coincidence of regular and rational languages in free monoids has been extended in many ways. In his seminal paper [17], Schützenberger generalized it to the realm of weighted automata, their behaviors, and rational formal power series. Weighted automata are classical non-deterministic automata in which transitions carry weights, which may model, e.g., the amount of resources needed for executing the transition or the probability of its successful execution. As these weights can be taken from any semiring, weighted automata have a rich structure theory, cf. [1, 8]. For most of the theory developed so far, it is crucial that in semirings multiplication distributes over addition. However, in non-classical kinds of logic, e.g., multi-valued logic [12] and quantum logic [2], the truth values can be modeled by bounded lattices, and the definition of such algebraic structures does not provide a distributivity law. Thus, Droste, Stüber, and Vogler [9] investigated weighted automata with weights in strong bimonoids, which generalize bounded lattices, and can be viewed as “semirings without distributivity”. Recently, Droste and Vogler [10] proved the coincidence of recognizable and rational formal power series under the assumption of bi-local finiteness, i.e. the local finiteness of both operations. Furthermore, they showed that strong bimonoids form a weight structure occurring naturally in different kinds of weighted automata, studied lately, e.g., in [3, 4].
On the other hand, Mazurkiewicz [14] introduced trace monoids as a model for the behavior of concurrent systems, cf. [5, 6]. They can be regarded as free partially commutative monoids where some generators may commute whenever the represented actions can occur independently in a given system. In general, the recognizable trace languages are properly contained in the rational ones, and by Ochmański’s theorem [16] they coincide with the c-rational trace languages where the iteration is restricted to connected languages. Similar to Schützenberger’s approach, Droste and Gastin [7] extended this result to a situation with weights from a commutative semiring. They introduced the concepts of c-rational and mc-rational trace series, where for the latter the iteration is further restricted to mono-alphabetic series, and showed the coincidence of recognizable and mc-rational series, and their coincidence with c-rational series if the semiring is also additively idempotent [7]. Here, we want give a joint extension of the results of Ochmański, Droste and Gastin, and Droste and Vogler to trace series with weights in bi-locally finite strong bimonoids. However, we need new proof strategies since the techniques of Droste and Gastin crucially depend on the distributivity of the semiring and cannot be adopted. Moreover, the automaton model of Droste and Gastin turned out to be very useful for investigating the relationship between other formalisms describing trace series over semirings, cf. [13, 15, 11]. This further motivates studying weighted trace automata over strong bimonoids and their behavior.

The main results of this paper are the following. First, we introduce a model for weighted trace automata with weights in a strong bimonoid. This task is non-trivial since the model of Droste and Gastin has no well-defined semantics in the absence of distributivity. Second, we prove the coincidence of recognizable and mc-rational series under the assumption of bi-local finiteness. In contrast to the result of Droste and Gastin, we need no commutativity of the weight structure. Third, we show their common coincidence with c-rational series if the bimonoid is idempotent, commutative, and bi-locally finite. Finally, we prove that we can neither drop idempotence nor commutativity from the assumptions of this result.

2. Basic concepts: traces and bimonoids

Here we recall the necessary notation and background of trace theory and strong bimonoids. For more details, we refer the reader to [5, 6, 9, 10].

An independence alphabet is a pair $(\Sigma, I)$ consisting of a finite non-empty set $\Sigma$ and an irreflexive symmetric relation $I$ on $\Sigma$, called independence relation. The complement $D = (\Sigma \times \Sigma) \setminus I$ of $I$ is called dependence relation. The congruence $\sim$ on $\Sigma^*$ generated by $\{(ab, ba) \mid (a, b) \in I\}$ is called trace equivalence. The quotient monoid $M = M(\Sigma, I) := \Sigma^*/\sim$ is the trace monoid over $(\Sigma, I)$ and its elements are called traces. For $w \in \Sigma^*$ let $[w]$ be the equivalence class of $w$ in $M$, the empty trace $[\varepsilon]$ is also denoted by $1_M$. Moreover, we let $\varphi : \Sigma^* \to M$ always be the canonical epimorphism. As usual, a trace language $L \subseteq M$ is called recognizable if there exists a morphism $h : M \to S$ into a finite monoid $S$ and a subset $F \subseteq S$ such that $L = h^{-1}(F)$. Recognizable trace languages can be characterized by their preimage under $\varphi$:

**Proposition 2.1.** Let $L \subseteq M$ be a trace language. Then $L$ is recognizable iff $\varphi^{-1}(L)$ is recognizable.
It is well known that there are rational trace languages which are not recognizable. This leads to the concept of \(c\)-rationality. Let \(\text{alph}(w)\) be the set of all letters occurring in \(w \in \Sigma^*\), called the alphabet of \(w\). Since trace equivalent words have the same alphabet, we may put \(\text{alph}(|w|) = \text{alph}(w)\). A subset \(X \subseteq \Sigma\) is called connected (with respect to \(D\)) if there are no non-empty sets \(A, B \subseteq \Sigma\) satisfying \(X = A \cup B\) and \(A \times B \subseteq I\). A trace \(t \in M\) is connected if \(\text{alph}(t)\) is connected; a language \(L \subseteq M\) is connected if all \(t \in L\) are connected. Furthermore, \(L \subseteq M\) is called \(c\)-rational if it can be constructed from the finite languages in \(M\) using union, product, and iteration, where the latter is applied only to connected languages. This notion admits a characterization of the recognizable trace languages similar to Kleene’s theorem:

**Theorem 2.2** (Ochmański [16]). Let \(L \subseteq M\) be a trace language. Then \(L\) is recognizable iff \(L\) is \(c\)-rational.

A strong bimonoid is an algebraic structure \(A = (A, +, \cdot, 0, 1)\) such that \((A, +, 0)\) is commutative monoid, \((A, \cdot, 1)\) is a monoid, and 0 is an absorbing element, i.e. \(0 \cdot a = a \cdot 0 = 0\) for all \(a \in A\). The strong bimonoid is called commutative if \((A, \cdot, 1)\) is commutative, idempotent if \((A, +, 0)\) is idempotent, additively locally finite (resp. multiplicatively locally finite) if all finitely generated submonoids of \((A, +)\) (resp. \((A, \cdot)\)) are finite, bi-locally finite if \(A\) is additively and multiplicatively locally finite, and distributive if multiplication distributes over addition. A semiring is a distributive strong bimonoid.

Important examples for strong bimonoids include all semirings and bounded lattices; note that the latter are commutative, idempotent, and bi-locally finite. For each \(0 < \delta < 1\) the structure \((\{0\} \cup [\delta, 1], +, \otimes, 0, 1)\) with \(a + b = \min\{a + b, 1\}\) and \(a \otimes b = ab\) if \(ab \notin (0, \delta)\) and \(a \otimes b = 0\) otherwise is a commutative, non-distributive, bi-locally finite strong bimonoid. It models the real numbers with bounded precision of addition and multiplication. For a range of further examples of strong bimonoids which are not semirings we refer the reader to [9, 10].

For the rest of this paper, we fix a trace monoid \(M = M(\Sigma, I)\), the canonical epimorphism \(\varphi : \Sigma^* \to M\), and a strong bimonoid \(A\).

### 3. Weighted trace automata and rational expressions

#### 3.1. Weighted trace automata

A (formal) trace series over \(A\) and \(M\) is a mapping \(S : M \to A\). It is often written as a formal sum \(S = \sum_{t \in M} (S, t)\) where \((S, t) = S(t)\). Mappings \(\Sigma^* \to A\) are called (formal) word series. The collection of all trace series over \(A\) and \(M\) is denoted by \(A\langle\langle M\rangle\rangle\), and similarly, \(A\langle\langle \Sigma^*\rangle\rangle\) is defined. A weighted (word) automaton over \(A\) and \(\Sigma\) is a 4-tuple \(\mathcal{A} = (Q, \text{in}, \text{wt}, \text{out})\) such that \(Q\) is a finite set (of states), \(\text{in}, \text{out} : Q \to A\) are the initial resp. final weight function and \(\text{wt} : Q \times \Sigma \times Q \to A\) is the transition weight function. The (word) behavior of \(\mathcal{A}\) is the word series \(\|\mathcal{A}\|_W \in A\langle\langle \Sigma^*\rangle\rangle\) with

\[
\|\mathcal{A}\|_W, \sigma_1 \ldots \sigma_n = \sum_{(q_0, \ldots, q_n) \in Q^{n+1}} \text{in}(q_0) \cdot \text{wt}(q_0, \sigma_1, q_1) \cdot \ldots \cdot \text{wt}(q_{n-1}, \sigma_n, q_n) \cdot \text{out}(q_n). \tag{1}
\]
A word series $S \in A\langle \Sigma^* \rangle$ is called *recognizable* if there is a weighted word automaton $A$ such that $S = \|A\|_W$.

In order to define a model for weighted trace automata based on weighted word automata we want to restrict the class of automata permitted such that for all $u, v \in \Sigma^*$ with $u \sim v$ we have $(\|A\|_W, u) = (\|A\|_W, v)$. This is captured by the following definition and proposition.

**Definition 3.1.** A weighted word automaton $A$ has the *I-diamond-property* if for all $(\sigma, \tau) \in I$ and $p, r \in Q$ exists bijection $f = f_{p,r}^{\sigma,\tau} : Q \to Q$ such that for any $q \in Q$ we have

$$\text{wt}(p, \sigma, q) \cdot \text{wt}(q, \tau, r) = \text{wt}(p, \tau, f(q)) \cdot \text{wt}(f(q), \sigma, r).$$

**Proposition 3.2.** Let $A$ be a weighted word automaton having the I-diamond-property. Then $(\|A\|_W, u) = (\|A\|_W, v)$ for all $u, v \in \Sigma^*$ with $u \sim v$.

**Proof.** Due to the definition of $\sim$ it suffices to consider $u = x\sigma\tau y$ and $v = x\tau\sigma y$ where $x, y \in \Sigma^*$ and $(\sigma, \tau) \in I$. Let $k = |x\sigma|$ and $n = |u|$. Then the mapping $f : Q^{n+1} \to Q^{n+1}$ replacing $q_k$ in $(q_0, \ldots, q_n)$ by $f_{q_{k-1},q_{k+1}}^{\sigma,\tau}(q_k)$ is bijection such that in (1) the summand for $(q_0, \ldots, q_n)$ and $u$ equals the summand for $f(q_0, \ldots, q_n)$ and $v$. Thus, $(\|A\|_W, u) = (\|A\|_W, v)$.

**Definition 3.3.** A *weighted trace automaton* is a weighted word automaton $A$ having the I-diamond-property. Its (trace) behavior is the trace series $\|A\|_T \in A\langle \mathbb{M} \rangle$ defined by

$$(\|A\|_T, [w]) = (\|A\|_W, w).$$

A trace series $S \in A\langle \mathbb{M} \rangle$ is called *recognizable* if there exists a weighted trace automaton such that $S = \|A\|_T$.

**Remark 3.4.** From Definition 3.3 and Theorem 4.1 in [13] we can conclude that for commutative semirings this notion of recognizability coincides with that of Droste and Gastin in [7].

### 3.2. Rational expressions

The goal of this paper is to describe the behavior of weighted trace automata using weighted rational expressions. For this, we introduce some notation. For a trace series $S \in A\langle \mathbb{M} \rangle$ we call the set $\text{supp}(S) = \{ t \in \mathbb{M} \mid (S, t) \neq 0 \}$ the support of $S$. A *polynomial* is a trace series with finite support. For $S, T \in A\langle \mathbb{M} \rangle$ and $a \in A$ we define new trace series $a \cdot S, S + T, S \cdot T \in A\langle \mathbb{M} \rangle$ called exterior product, sum and Cauchy product by letting

$$(a \cdot S, t) = a \cdot (S, t), \quad (S + T, t) = (S, t) + (T, t), \quad \text{and} \quad (S \cdot T, t) = \sum_{t = u_{i\cdots u_n}} (S, u_1) \cdots (S, u_n).$$

Since the Cauchy product is associative iff $A$ is distributive, powers of $S$ need to be considered explicitly. For $n \in \mathbb{N}$ define $S^n \in A\langle \mathbb{M} \rangle$ as

$$(S^n, t) = \sum_{t = u_1 \cdots u_n} (S, u_1) \cdots (S, u_n).$$
If $S$ is proper, i.e. $(S,1_M) = 0$, we define the iteration $S^* \in A\langle\langle M\rangle\rangle$ of $S$ as
\[
(S^*, t) = \sum_{n \in \mathbb{N}} (S^n, t).
\]
A trace series $S \in A\langle\langle M\rangle\rangle$ is called rational if it can be constructed from the polynomials using sum, Cauchy product, and iteration, where the latter is applied only to proper series.

For the case of word series over semirings, Schützenberger’s theorem states the equivalence of recognizability and rationality:

**Theorem 3.5** (Schützenberger [17]). Let $K$ be a semiring and $S \in K\langle\langle \Sigma^*\rangle\rangle$ a word series. Then $S$ is recognizable iff $S$ is rational.

As observed in [7], this theorem cannot be generalized directly to trace series due to problems concerning connectedness. A trace series $S \in A\langle\langle M\rangle\rangle$ is called connected if $\text{supp}(S)$ is connected, and mono-alphabetic if $\text{alph}(t) = \text{alph}(t')$ for all $t,t' \in \text{supp}(S)$. We call $S$ c-rational if it can be constructed from the polynomials using sum, Cauchy product, and iteration, where the latter is applied only to proper, connected series. If the iteration is further restricted to proper, mono-alphabetic, connected series, we call $S$ mc-rational.

**Theorem 3.6** (Droste and Gastin [7]). Let $K$ be a semiring and $S \in K\langle\langle M\rangle\rangle$ a trace series.

1. If $S$ is recognizable, then $S$ is mc-rational.
2. If $K$ is commutative and $S$ is mc-rational, then $S$ is recognizable.
3. If $K$ is idempotent and commutative and $S$ is c-rational, then $S$ is recognizable.

Schützenberger’s theorem was also extended to bi-locally finite strong bimonoids:

**Theorem 3.7** (Droste and Vogler [10]). Let $A$ be a bi-locally finite strong bimonoid and $S \in A\langle\langle \Sigma^*\rangle\rangle$ a word series. Then $S$ is recognizable iff $S$ is rational.

The main result of this paper is a joint extension of the three previous theorems:

**Theorem 3.8.** Let $A$ be a bi-locally finite strong bimonoid and $S \in A\langle\langle M\rangle\rangle$ a trace series.

1. $S$ is recognizable iff $S$ is mc-rational.
2. If $A$ is idempotent and commutative, then $S$ is recognizable iff $S$ is c-rational.

Since all bounded lattices are commutative, idempotent, bi-locally finite strong bimonoids, we obtain the following corollary:
Corollary 3.9. Let $\mathcal{L}$ be a bounded lattice and $S \in \mathcal{L}\langle\langle M\rangle\rangle$ a trace series. The following are equivalent:

1. $S$ is recognizable,
2. $S$ is mc-rational,
3. $S$ is c-rational.

In order to prove the main theorem we need another concept called recognizable step series.

3.3. Recognizable step series

For a language $L \subseteq M$ we define its characteristic series $1_L \in A\langle\langle M\rangle\rangle$ by

$$(1_L, t) = \begin{cases} 1 & \text{if } t \in L, \\ 0 & \text{if } t \not\in L. \end{cases}$$

A trace series $S \in A\langle\langle M\rangle\rangle$ is called recognizable step series if there are $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$ and recognizable trace languages $L_1, \ldots, L_n \subseteq M$ such that

$$S = \sum_{i=1}^{n} a_i \cdot 1_{L_i}. \quad (2)$$

It is easy to see that $S$ is a recognizable step series if and only if $\text{im}(S)$ is finite and $S^{-1}(a)$ is recognizable for any $a \in \text{im}(S)$. We will often use this as a characterization of recognizable step series without mentioning it explicitly. In the situation of equation (2) we can consider the minimal automaton of the recognizable word language $\varphi^{-1}(L_i)$ as a weighted word automaton with initial and transition weights 0 and 1 and final weights 0 and $a_i$, where the latter is assigned to a state iff it is final. The disjoint union of all these automata is a weighted trace automaton with behavior $S$. Hence, all recognizable step series are recognizable. We will show the converse of this statement under the additional assumption of bi-local finiteness. The following theorem generalizes a result of Droste, Stüber, and Vogler [9] which states the word case:

Theorem 3.10. Let $A$ be a bi-locally finite strong bimonoid and $S \in A\langle\langle M\rangle\rangle$ a recognizable trace series. Then $S$ is a recognizable step series.

In order to prove this theorem we need another notion. For a trace series $S \in A\langle\langle M\rangle\rangle$ we define a word series $\varphi^{-1}(S) \in A\langle\langle \Sigma^*\rangle\rangle$ by letting

$$(\varphi^{-1}(S), w) = (S, [w]).$$

From the characterization of recognizable step series above and Proposition 2.1 we can conclude that $S$ is a recognizable step series if and only if $\varphi^{-1}(S)$ is a recognizable step series.
Proof of Theorem 3.10. Let $A$ be a weighted trace automaton with $\|A\|_T = S$. If we consider the underlying weighted word automaton, we obtain $\|A\|_W = \varphi^{-1}(S)$, and hence $\varphi^{-1}(S)$ is recognizable. By the word case of Theorem 3.10 due to [9], $\varphi^{-1}(S)$ is a recognizable step series. It follows that $S$ is also a recognizable step series.

This result turns out to be very important for the proof of Theorem 3.8, since it allows us to consider recognizable step series instead of recognizable trace series.

4. Coincidence of recognizability and mc-rationality

4.1. Recognizable step series are mc-rational

In this section we prove the following theorem:

Theorem 4.1. Let $A$ be a strong bimonoid and $S \in A\langle\langle M \rangle\rangle$ a recognizable step series. Then $S$ is mc-rational.

Since mc-rationality implies c-rationality by definition, this also shows that recognizable step series are c-rational. In order to prove the theorem, we need the following technical lemma:

Lemma 4.2. Let $A$ be a strong bimonoid and $L \subseteq M$ a trace language. If $1_L \in N\langle\langle M \rangle\rangle$ is mc-rational, then $1_L \in A\langle\langle M \rangle\rangle$ is also mc-rational.

Proof (sketch). Let $S = 1_L \in N\langle\langle M \rangle\rangle$ and $T = 1_L \in A\langle\langle M \rangle\rangle$. We prove the claim by induction on the mc-rational construction of $S$. Clearly, if $S$ is a polynomial, $T$ is also a polynomial. Now, assume $S = S_1 + S_2$ for mc-rational $S_1, S_2 \in N\langle\langle M \rangle\rangle$. If there was a $t \in M$ such that $(S_1, t) > 1$ or $(S_2, t) > 1$, we would get $(S, t) > 1$, which is impossible. Hence, there are $L_1, L_2 \subseteq M$ such that $L = L_1 \cup L_2$ and $S_i = 1_{L_i}$ for $i = 1, 2$. Moreover, $L_1$ and $L_2$ are disjoint, since $t \in L_1 \cap L_2$ would imply $(S, t) = 2$. By induction, $T_1 = 1_{L_1}, T_2 = 1_{L_2} \in A\langle\langle M \rangle\rangle$ are mc-rational and from $T = T_1 + T_2$ we conclude that $T$ is also mc-rational. The remaining cases work similarly.

Proof of Theorem 4.1. Consider $a_1, \ldots, a_n \in A$ and recognizable languages $L_1, \ldots, L_n \subseteq M$ such that $S = \sum_{i=1}^n a_i \cdot 1_{L_i}$. Then $1_{L_i} \in N\langle\langle M \rangle\rangle$ is a recognizable step series, hence recognizable as mentioned above, for each $i = 1, \ldots, n$. Thus, by Theorem 3.6 and Lemma 4.2, $1_{L_i} \in A\langle\langle M \rangle\rangle$ is mc-rational. Moreover, since $a_i \cdot 1_{\{1_M\}}$ is a polynomial and $a_i \cdot 1_{L_i} = (a_i \cdot 1_{\{1_M\}}) \cdot 1_{L_i}$, $S$ is a finite sum of mc-rational series, and hence itself mc-rational.

4.2. mc-rational trace series are recognizable step series

Next, we show the following:

Theorem 4.3. Let $A$ be a bi-locally finite strong bimonoid and $S \in A\langle\langle M \rangle\rangle$ an mc-rational trace series. Then $S$ is a recognizable step series.
We prove this result by induction on the mc-rational construction of \( S \). For polynomials \( S \in A\langle\langle M\rangle\rangle \) the claim follows from the finiteness of \( \text{supp}(S) \) and the sum

\[
S = \sum_{t \in \text{supp}(S)} (S, t) \cdot \mathbb{1}_{\{t\}}.
\]

Closure of the class of recognizable step series under sum is obvious. In order to prove closure under Cauchy product, we need the following proposition which is implicit in [7]:

**Proposition 4.4 ([7]).** Let \( K \) be a commutative semiring and \( S \in K\langle\langle M\rangle\rangle \) a trace series. Then \( S \) is recognizable iff \( \varphi^{-1}(S) \) is recognizable.

The word case of the following lemma is contained in section 3.2 of [1]:

**Lemma 4.5.** Let \( S \in \mathbb{N}\langle\langle M\rangle\rangle \) be a recognizable trace series. Then for all \( k, \ell \in \mathbb{N} \) the trace language \( S^{-1}(k + \ell \cdot \mathbb{N}) \) is recognizable.

**Proof.** Due to Proposition 4.4 the word series \( \varphi^{-1}(S) \) is recognizable. By the word case, \( (\varphi^{-1}(S))^{-1}(k + \ell \cdot \mathbb{N}) = \varphi^{-1}(S^{-1}(k + \ell \cdot \mathbb{N})) \) is a recognizable word language, and the claim follows from Proposition 2.1.

Now, we are in a position to prove the following:

**Proposition 4.6.** Let \( A \) be an additively locally finite strong bimonoid and \( S, T \in A\langle\langle M\rangle\rangle \) recognizable step series. Then \( S \cdot T \) is also a recognizable step series.

**Proof.** From the definition \( (S \cdot T, t) = \sum_{u,v} (S, u) \cdot (T, v) \) we observe that all possible summands are of the form \( a \cdot b \) with \( a \in \text{im}(S) \) and \( b \in \text{im}(T) \). Thus, the main idea is to count for each product \( a \cdot b \) how often it occurs. This is accomplished by the mapping \( C_{a,b} : \mathbb{M} \rightarrow \mathbb{N} \) defined as

\[
C_{a,b}(t) = \text{card} \{ (u, v) \mid t = uv, (S, u) = a, (T, v) = b \}.
\]

Moreover, let \( \eta_{a,b} : (\mathbb{N}, +) \rightarrow (A, +) \) be the unique monoid morphism satisfying \( \eta_{a,b}(1) = a \cdot b \). Using the composed mapping \( U_{a,b} = \eta_{a,b} \circ C_{a,b} : \mathbb{M} \rightarrow A \), we can express \( S \cdot T \) as

\[
S \cdot T = \sum_{a \in \text{im}(S)} \sum_{b \in \text{im}(T)} U_{a,b}.
\]

Since \( S \) and \( T \) are recognizable step series their images are finite. Thus, it suffices to show that \( U_{a,b} \in A\langle\langle M\rangle\rangle \) is a recognizable step series for all \( a, b \in A \). Clearly, \( \text{im}(U_{a,b}) \) is finite since \( \text{im}(U_{a,b}) \subseteq \text{im}(\eta_{a,b}) \) and \((A, +)\) is locally finite. Now, consider \( c \in \text{im}(U_{a,b}) \). Since \( \eta_{a,b} \) is a monoid morphism from \((\mathbb{N}, +)\) to \( A \) with finite image and \( c \in \text{im}(\eta_{a,b}) \), there are \( k, \ell \in \mathbb{N} \) such that \( \eta_{a,b}^{-1}(c) = k + \ell \cdot \mathbb{N} \). Thus,

\[
U_{a,b}^{-1}(c) = C_{a,b}^{-1}(k + \ell \cdot \mathbb{N}).
\]
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Considering the series $S^{-1}(a), T^{-1}(b) \in N\langle\langle M\rangle\rangle$, from the definition of $C_{a,b}$ follows immediately

$$C_{a,b} = S^{-1}(a) \cdot T^{-1}(b).$$

Since $S$ and $T$ are recognizable step series, $S^{-1}(a)$ and $T^{-1}(b)$ are recognizable trace languages. Thus, $S^{-1}(a), T^{-1}(b) \in N\langle\langle M\rangle\rangle$ are recognizable trace series. By Theorem 3.6 their Cauchy product $C_{a,b}$ is also recognizable, and by Lemma 4.5, $U_{a,b}^{-1}(c)$ is a recognizable language.

Next, we prove the closure under iteration of proper, mono-alphabetic, connected series. This uses a similar but more involved technique.

**Proposition 4.7.** Let $A$ be a bi-locally finite strong bimonoid and $S \in A\langle\langle M\rangle\rangle$ a proper, mono-alphabetic, connected recognizable step series. Then $S^*$ is also a recognizable step series.

**Proof (sketch).** Recall that

$$(S^*,t) = \sum_{n \in N, \sum_{i=1}^{n} u_i = t} (S,u_1) \cdots (S,u_n).$$

We define a map $C_a : M \rightarrow N$ for counting occurrences of addends $a \in A$ as follows

$$C_a(t) = \text{card} \{ (u_1, \ldots, u_n) \mid t = u_1 \cdots u_n, u_i \in \text{supp}(S), (S,u_1) \cdots \cdot (S,u_n) = a \}.$$

Let $B = \text{im}(S) \setminus \{0\}$ and $\eta_a : (N,+) \rightarrow (A,\cdot)$ resp. $\psi : B^* \rightarrow (A,\cdot)$ be the unique monoid morphisms with $\eta_a(1) = a$ resp. $\psi(b) = b$ for all $b \in B$. Using $\eta_a \circ C_a : M \rightarrow A$ we can express $S^*$ as

$$S^* = \sum_{a \in \text{im}(\psi)} \eta_a \circ C_a.$$

This sum is finite since $(A,\cdot)$ is locally finite. As in the previous proof and using that $(A,\cdot)$ is also locally finite, it suffices to show that $C_a \in N\langle\langle M\rangle\rangle$ is recognizable.

For this, the main idea is to consider the word language $\psi^{-1}(a)$, which is recognizable since $(A,\cdot)$ is locally finite. We take a rational expression for this language and replace every letter $b$ in this expression by an mc-rational expression for $1_{S^{-1}(b)} \in N\langle\langle M\rangle\rangle$. The result will be a rational expression for $C_a \in N\langle\langle M\rangle\rangle$. Using the fact that $S$ is mono-alphabetic and connected, we can show that this expression is also mc-rational. Hence, by Theorem 3.6, $C_a$ is recognizable.

5. Coincidence of recognizability and c-rationality

5.1. C-rational trace series are recognizable step series

This section is devoted to the proof of the following result:

**Theorem 5.1.** Let $A$ be an idempotent, commutative, bi-locally finite strong bimonoid and $S \in A\langle\langle M\rangle\rangle$ a c-rational trace series. Then $S$ is a recognizable step series.
Again, we prove this result by induction on the mc-rational construction of $S$. It was shown in the last section that the class of recognizable step series contains the polynomials, and is closed under sum and Cauchy product. It remains to prove the closure under iteration of proper, connected series.

**Proposition 5.2.** Let $A$ be an idempotent, commutative, bi-locally finite strong bimonoid and $S \in A\langle\langle M\rangle\rangle$ a proper, connected recognizable step series. Then $S^*$ is also a recognizable step series.

**Proof (sketch).** Since $A$ is idempotent, in contrast to the proof Proposition 4.7, we do not need to count occurrences of $a \in A$ in the sum defining $(S^*,t)$ for a given trace $t \in M$. It only matters if $a$ occurs at least once. Therefore, we define a trace language $E_a \subseteq M$ as

$$E_a = \{ t \in M \mid \exists u_1, \ldots, u_n \in \text{supp}(S) : t = u_1 \cdots u_n, (S,u_1) \cdot \ldots \cdot (S,u_n) = a \}$$

Let $B = \text{im}(S) \setminus \{0\}$ and $C$ denote the submonoid of $(A,\cdot)$ generated by $B$. Then we obtain

$$S^* = \sum_{a \in C} a \cdot 1_{E_a}.$$ 

Since $C$ is finite, it suffices to show that $E_a$ is recognizable for any $a \in C$. We note that we cannot adopt directly the proof strategy for Proposition 4.7 due to problems with the connectedness. However, using an auxiliary construction we can circumvent these issues. Therefore, we define an independence alphabet $(\Gamma, H)$ by letting

$$\Gamma = \{(b,X) \mid b \in B, X \subseteq \Sigma \text{ is non-empty and connected}\} \quad \text{and} \quad ((b,X),(c,Y)) \in H \iff X \times Y \subseteq I \wedge (b,X) \neq (c,Y).$$

Since $A$ is commutative, the projection $\Gamma \to C, (b,X) \mapsto b$ induces a monoid morphism $\psi : M(\Gamma, H) \to C$. Now, the main idea is to take a $c$-rational expression for the recognizable trace language $\psi^{-1}(a)$ and to replace every letter $(b,X)$ by a $c$-rational expression for the recognizable language $L_{(b,X)} = \{ t \in M \mid (S,t) = b, \text{alph}(t) = X \}$. This substitution is well defined since the construction ensures that $L_\beta \cdot L_\gamma = L_\gamma \cdot L_\beta$ whenever $(\beta,\gamma) \in H$. The result is a rational expression for $E_a$. Due to the choice of $(\Gamma, H)$ and using that $L_{(b,X)}$ is connected and mono-alphabetic with alphabet $X$, we can show that the expression obtained is also $c$-rational, and hence $E_a$ is recognizable by Theorem 2.2.

**5.2. Sharpness of the result**

Finally, we give two examples showing that we can neither drop idempotence nor commutativity from the assumptions of Theorem 5.1 and Proposition 5.2. For both examples let $M$ be the trace monoid $M = M(\Sigma,I)$ with $\Sigma = \{ \sigma, \tau \}$ and $\sigma I \tau$.

**Example 5.3.** Let $p$ be a prime number and $\mathbb{F}_p$ be the field with $p$ elements. Consider the series

$$S = [\sigma] + [\tau] \in \mathbb{F}_p\langle\langle M\rangle\rangle.$$
Then $S^*$ is obviously $c$-rational. Using the canonical identification $\mathbb{M} \cong \mathbb{N} \times \mathbb{N}$ we have

$$(S^*, (m, n)) = \begin{pmatrix} m + n \\ m \end{pmatrix} \mod p.$$  

Thus, $L = (S^*)^{-1}(0)$ is the set of all $(m, n)$ such that $\begin{pmatrix} m + n \\ m \end{pmatrix}$ is divisible by $p$. By counting prime factors $p$ we obtain $(p^k, (p - 1) \cdot p^\ell) \in L$ iff $k = \ell$, for all $k, \ell \in \mathbb{N}$. For $k \neq \ell$ we get

$$(p^k, 0) + (0, (p - 1) \cdot p^k) \in L \quad \text{whereas} \quad (p^\ell, 0) + (0, (p - 1) \cdot p^k) \notin L,$$

i.e. $(p^k, 0)$ and $(p^\ell, 0)$ are syntactically incongruent. Hence, $L$ is not recognizable and $S^*$ is not a recognizable step series.

**Example 5.4.** Let $(A, \cdot)$ be the syntactic monoid of the recognizable word language $L = (\alpha\beta)^* \subseteq \{\alpha, \beta\}^*$. We denote its five elements as follows:

$$0 = [\alpha\alpha]_L = [\beta\beta]_L, \quad a = [\alpha]_L, \quad b = [\beta]_L, \quad c = [\beta\alpha]_L, \quad \text{and} \quad 1 = [\varepsilon]_L = [\alpha\beta]_L.$$  

Notice that 0 acts as an absorbing element. Consider the linear order $0 < a < b < c < 1$. Then $(A, \max, \cdot, 0, 1)$ becomes an idempotent, finite strong bimonoid. Clearly, the trace series

$$S = a \cdot [\sigma] + b \cdot [\tau] \in A\langle\langle M\rangle\rangle$$  

is a proper and connected polynomial, and hence $S^*$ is $c$-rational. All possible factorizations of 1 in $A$ using only factors from $\text{im}(S) = \{a, b\}$ are of the form $1 = abab \cdots ab$. Hence, for a trace $t \in \mathbb{M}$ with $(S^*, t) = 1$ there must be a decomposition of the form $t = [\sigma][\tau] \cdots [\sigma][\tau]$. Conversely, if such a decomposition exists, then $(S^*, t) = 1$. Thus, we obtain

$$(S^*)^{-1}(1) = ([\sigma][\tau])^*.$$  

However, this trace language is not recognizable, since its preimage under $\varphi$ is the set of all words $w \in \Sigma^*$ containing as many $\sigma$’s as $\tau$’s. Hence, $S^*$ is not a recognizable step series.

### 6. Conclusions

We showed that for bi-locally finite strong bimonoids any trace series recognizable by a weighted trace automaton can also be described by an $mc$-rational expression, and vice versa. If the bimonoid is also commutative and idempotent, the same holds for $c$-rational expressions. Moreover, all constructions and the corresponding translations between automata and expressions are effective. However, the proofs showed that algorithms dealing with weighted trace automata should better use their representation as recognizable step series. The automaton model of Droste and Gastin was very useful for proving that weighted logics and weighted asynchronous cellular automata over semirings are equally expressive, cf. [13, 15, 11]. Hopefully, the model introduced in this paper will play the same important role for the corresponding relationship in the case of weights in a strong bimonoid.
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References


