A Kleene-Schützenberger Theorem for Trace Series over Bounded Lattices

Martin Huschenbett
Institut für Theoretische Informatik
Technische Universität Ilmenau
martin.huschenbett@tu-ilmenau.de

Abstract. We study weighted trace automata with weights in strong bimonoids. Traces form a generalization of words that allow to model concurrency; strong bimonoids are algebraic structures that can be regarded as “semirings without distributivity”. A very important example for the latter are bounded lattices, especially non-distributive ones. We show that if both operations of the bimonoid are locally finite, then the classes of recognizable and mc-rational trace series coincide and, in general, are properly contained in the class of c-rational series. Moreover, if, in addition, in the bimonoid the addition is idempotent and the multiplication is commutative, then all three classes coincide.

Keywords: weighted automata, concurrency, multi-valued logics, traces, formal power series

1. Introduction

In the theory of automata and formal languages, Kleene’s foundational theorem [14] on the coincidence of regular and rational languages in free monoids has been extended in many ways. In his seminal paper [21], Schützenberger generalized it to the realm of weighted automata, their behaviors, and rational formal power series. Weighted automata are classical non-deterministic automata whose transitions carry weights, which may model, e.g., the amount of resources needed for executing the transition or the probability of its successful execution. As these weights can be taken from any semiring, weighted automata have a rich structure theory, cf. [1, 8]. For most of the theory developed so far, it is crucial that in semirings multiplication distributes over addition. However, in non-classical kinds of logic, e.g., multi-valued
logic [16] and quantum logic [2], the truth values can be modeled by bounded lattices, and the definition of such algebraic structures does not provide a distributivity law. Thus, Droste, Stüber, and Vogler [9] investigated weighted automata with weights in strong bimonoids, which generalize bounded lattices, and can be viewed as “semirings without distributivity”. Recently, Droste and Vogler [10] proved the coincidence of recognizable and rational formal power series under the assumption of bi-local finiteness, i.e. the local finiteness of both operations. Furthermore, they showed that strong bimonoids form a weight structure occurring naturally in different kinds of weighted automata, studied recently, e.g., in [3, 4].

On the other hand, Mazurkiewicz [18] introduced trace monoids as a model for the behavior of concurrent systems, cf. [5, 6]. They can be regarded as free partially commutative monoids where some generators may commute whenever the represented actions can occur independently in a given system. In general, the recognizable trace languages are properly contained in the rational ones, and by Ochmański’s theorem [20] they coincide with the c-rational trace languages where the iteration is restricted to connected languages. Similar to Schützenberger’s approach, Droste and Gastin [7] extended this result to a situation with weights from a commutative semiring. They introduced the concepts of c-rational and mc-rational trace series, where for the latter the iteration is further restricted to mono-alphabetic series, and showed the coincidence of recognizable and mc-rational series, and their coincidence with c-rational series if the semiring is also additively idempotent [7]. Here, we want to give a joint extension of the results of Ochmański, Droste and Gastin, and Droste and Vogler to trace series with weights in bi-locally finite strong bimonoids. However, we need new proof strategies since the techniques of Droste and Gastin crucially depend on the distributivity of the semiring and cannot be adopted. Moreover, the automaton model of Droste and Gastin turned out to be very useful for investigating the relationship between other formalisms describing trace series over semirings, cf. [17, 19, 12]. This further motivates studying weighted trace automata over strong bimonoids and their behavior.

The main results of this paper are the following. First, we introduce a model for weighted trace automata with weights in a strong bimonoid. This task is non-trivial since the model of Droste and Gastin has no well-defined semantics in the absence of distributivity. Second, we prove the coincidence of recognizable and mc-rational series under the assumption of bi-local finiteness. In contrast to the result of Droste and Gastin, we need no commutativity of the weight structure. Third, we show their common coincidence with c-rational series if the bimonoid is idempotent, commutative, and bi-locally finite. Finally, we prove that we can neither drop idempotence nor commutativity from the assumptions of this result.

2. Basic concepts

Here we recall the necessary notation and background of trace theory and strong bimonoids. For more details, we refer the reader to [5, 6, 9, 10].

2.1. Traces

An independence alphabet is a pair \((\Sigma, I)\) consisting of a non-empty finite set \(\Sigma\) and an irreflexive and symmetric independence relation \(I\) on \(\Sigma\). The complement \(D = (\Sigma \times \Sigma) \setminus I\) of \(I\) is called dependence relation. The congruence \(\sim\) on \(\Sigma^*\) generated by \\{(ab, ba) \mid (a, b) \in I\\} is called trace equivalence. The quotient monoid \(\overline{M} = \overline{M}(\Sigma, I) := \Sigma^*/\sim\) is the trace monoid over \((\Sigma, I)\) and its elements are called traces. For \(w \in \Sigma^*\) let \([w]\) be the equivalence class of \(w\) in \(\overline{M}\), the empty trace \([e]\) is also denoted by \(1_{\overline{M}}\).
Moreover, we let \( \varphi: \Sigma^* \to \mathbb{M} \) always be the canonical epimorphism. If \( I = \emptyset \), then \( \sim \) is the identity relation on \( \Sigma^* \) and \( \varphi \) becomes an isomorphism. Thus, we can regard the free monoid \( \Sigma^* \) as a special case of a trace monoid.

As usual, a trace language \( L \subseteq \mathbb{M} \) is called recognizable if there exists a morphism \( h: \mathbb{M} \to S \) into a finite monoid \( S \) and a subset \( F \subseteq S \) such that \( L = h^{-1}(F) \). Recognizable trace languages can be characterized by their preimage under \( \varphi \), cf. [20]:

**Proposition 2.1.** Let \( L \subseteq \mathbb{M} \) be a trace language. Then \( L \) is recognizable if and only if \( \varphi^{-1}(L) \) is recognizable.

The concatenation \( L_1 \cdot L_2 \) of two trace languages \( L_1, L_2 \subseteq \mathbb{M} \) is defined as the set of all traces \( t_1 \cdot t_2 \) with \( t_1 \in L_1 \) and \( t_2 \in L_2 \). Similarly, the iteration \( L^* \) of \( L \subseteq \mathbb{M} \) is the set of all products \( t_1 \cdots t_n \) with \( n \in \mathbb{N} \) and \( t_1, \ldots, t_n \in L \). A trace language is said to be rational if it can be constructed from the finite subsets of \( \mathbb{M} \) using union, concatenation, and iteration. It is well known that every recognizable trace language is rational whereas the converse implication is, in general, incorrect. This leads to the concept of \( c \)-rationality.

Let \( \text{alph}(w) \) be the set of all letters occurring in \( w \in \Sigma^* \), called the alphabet of \( w \). Since trace equivalent words have the same alphabet, we may put \( \text{alph}([w]) = \text{alph}(w) \). A subset \( X \subseteq \Sigma \) is called connected (with respect to \( D \)) if there are no non-empty sets \( A, B \subseteq \Sigma \) satisfying \( X = A \cup B \) and \( A \times B \subseteq I \). A trace \( t \in \mathbb{M} \) is connected if \( \text{alph}(t) \) is connected; a language \( L \subseteq \mathbb{M} \) is connected if all \( t \in L \) are connected. Furthermore, \( L \subseteq \mathbb{M} \) is called \( c \)-rational if it can be constructed from the finite languages in \( \mathbb{M} \) using union, concatenation, and iteration, where the latter is applied only to connected languages. This notion admits a characterization of the recognizable trace languages similar to Kleene’s theorem:

**Theorem 2.2.** (Ochmański [20])
Let \( L \subseteq \mathbb{M} \) be a trace language. Then \( L \) is recognizable if and only if \( L \) is \( c \)-rational.

### 2.2. Strong bimonoids

A bimonoid is an algebraic structure \( A = (A, +, \cdot, 0, 1) \) such that \((A, +, 0)\) and \((A, \cdot, 1)\) are both monoids. We call \( A \) a strong bimonoid if, additionally, the operation \( + \) is commutative and \( 0 \) is an absorbing element, i.e., \( 0 \cdot a = a \cdot 0 = 0 \) for all \( a \in A \). A strong bimonoid is called commutative if \((A, \cdot, 1)\) is commutative, idempotent if \((A, +, 0)\) is idempotent, additively locally finite (resp. multiplicatively locally finite) if all finitely generated submonoids of \((A, +)\) (resp. \((A, \cdot)\)) are finite, bi-locally finite if \( A \) is additively and multiplicatively locally finite, and distributive if multiplication distributes over addition. A semiring is a distributive strong bimonoid.

Important examples for strong bimonoids include all semirings and bounded lattices; note that the latter are commutative, idempotent, and bi-locally finite. For each \( 0 < \delta < 1 \) the structure \( \mathcal{R}_\delta = \{0\} \cup [0, 1], \oplus, \otimes, 0, 1 \) with \( a \oplus b = \min\{a + b, 1\} \) and \( a \otimes b = ab \) if \( ab \geq \delta \) and \( a \otimes b = 0 \) otherwise is a commutative, non-distributive, and bi-locally finite strong bimonoid. It models two aspects of computer arithmetic: first, there is a largest number, and second, numbers cannot be arbitrary close to zero. For a range of further examples of strong bimonoids which are not semirings we refer the reader to [9, 10].
For the rest of this paper, we fix a trace monoid $M = M(\Sigma, I)$, the canonical epimorphism $\varphi: \Sigma^* \to M$, and a strong bimonoid $A$.

## 3. Weighted trace automata and rational expressions

### 3.1. Weighted trace automata

A trace series over $A$ and $M$ is a mapping $S: M \to A$. It is often written as a formal sum

$$S = \sum_{t \in M} (S, t) t$$

where $(S, t) = S(t)$. Mappings $\Sigma^* \to A$ are called word series. We can and will regard a word series also as a trace series $M(\Sigma, \emptyset) \to A$. The collection of all trace series over $A$ and $M$ is denoted by $A\langle\langle M\rangle\rangle$, and similarly, $A\langle\langle \Sigma^*\rangle\rangle$ is the set of all word series over $A$ and $\Sigma$. A weighted (word) automaton over $A$ and $\Sigma$ is a 4-tuple $A = (Q, \text{in}, \text{wt}, \text{out})$ where

- $Q$ is a finite set (of states),
- $\text{in}, \text{out}: Q \to A$ are the initial respectively final weight function, and
- $\text{wt}: Q \times \Sigma \times Q \to A$ is the transition weight function.

The (word) behavior of $A$ is the word series $\|A\|_w \in A\langle\langle \Sigma^*\rangle\rangle$ defined by

$$\|A\|_w, \sigma_1 \ldots \sigma_n = \sum_{(q_0, \ldots, q_{n+1}) \in Q^{n+1}} \text{in}(q_0) \cdot \text{wt}(q_0, \sigma_1, q_1) \cdots \text{wt}(q_{n-1}, \sigma_n, q_n) \cdot \text{out}(q_n). \quad (1)$$

A word series $S \in A\langle\langle \Sigma^*\rangle\rangle$ is called recognizable if there is a weighted word automaton $A$ such that $S = \|A\|_w$.

**Example 3.1.** We consider the weighted word automaton $A$ over $R_{1/16}$ and $\Sigma = \{\alpha, \beta\}$ which is shown in Figure 1. A transition $(q_i, \sigma, q_j)$ with weight $a$, i.e., $\text{wt}(q_i, \sigma, q_j) = a$, is visualized as $q_i \xrightarrow{\sigma,a} q_j$. Initial and final weights are depicted as $b \xrightarrow{} q_1$ and $q_4 \xrightarrow{} c$, meaning $\text{in}(q_1) = b$ and $\text{out}(q_4) = c$. For clarity, arrows corresponding to a weight of 0 are omitted. The behavior of $A$ on the words $\alpha \beta$ and $\beta \alpha$ is calculated as

$$\|A\|_w, \alpha \beta = (1 \otimes 1/6 \otimes 1/2 \otimes 1/2) \oplus (1 \otimes 1/2 \otimes 1 \otimes 1/2) = 1/4$$

and

$$\|A\|_w, \beta \alpha = (1 \otimes 1/3 \otimes 1/4 \otimes 1/2) \oplus (1 \otimes 2/3 \otimes 3/4 \otimes 1/2) = 1/4.$$

It is easy to see that all other words $w \in \Sigma^* \setminus \{\alpha \beta, \beta \alpha\}$ yield $\|A\|_w, w = 0$.

Next, we elaborate how we can use weighted word automata as recognizers for trace series. Hence, we are interested in weighted automata which assign the same weight to all representatives of a certain trace:
Definition 3.2. A word series $S \in A\langle \Sigma \rangle^*$ is trace closed (w.r.t. $I$) if for all $u,v \in \Sigma^*$ with $u \sim v$ we have

$$(S,u) = (S,v).$$

Example 3.3. The behavior of the weighted word automaton $A$ from Example 3.1 is trace closed w.r.t. $I = \{ (\alpha,\beta), (\beta,\alpha) \}$. However, using “weighted trace automaton” simply as a synonym for “trace closed weighted word automaton” is inconvenient since this would define our automaton model by a semantic restriction which is, in general, undecidable. This undecidability follows from the fact that the equivalence problem for weighted word automata over the tropical semiring $\mathbb{N} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ is undecidable:

Theorem 3.4. (Krob [15])

The following decision problem is undecidable:

Instance: An alphabet $\Sigma$ and two weighted word automata $A_1$ and $A_2$ over $\mathcal{N}$ and $\Sigma$.

Question: Does $\|A_1\|_w = \|A_2\|_w$ hold true?

Proposition 3.5. The following decision problem is undecidable:

Instance: An independence alphabet $(\Sigma, I)$ and a weighted word automaton $A$ over $\mathcal{N}$ and $\Sigma$.

Question: Is $A$ trace closed (w.r.t. $I$)?

Proof:

The main idea is to reduce the problem from Theorem 3.4 to our decision problem. Therefore, let $\Gamma$ be an alphabet and $A_i = (Q_i, \text{in}_i, \text{wt}_i, \text{out}_i)$ for $i = 1, 2$ be two weighted word automata over $\mathcal{N}$ and $\Gamma$. We assume without loss of generality that both automata are initially normalized, i.e., for both $i = 1, 2$ there is a unique state $q_i \in Q_i$ with $\text{in}_i(q_i) \neq \infty$ which additionally satisfies $\text{in}_i(q_i) = 0$, cf. [11].

Now, we let $\alpha, \beta \not\in \Gamma$ be two new letters and construct a weighted word automaton $A$ over $\mathcal{N}$ and $\Sigma = \{ \alpha, \beta \} \cup \Gamma$ as depicted in Figure 2: We take the disjoint union of $A_1$ and $A_2$ and add three states $p_0, p_1, p_2$ and four transitions $p_0 \overset{\alpha|0}{\rightarrow} p_1 \overset{\beta|0}{\rightarrow} q_1$ and $p_0 \overset{\beta|0}{\rightarrow} p_2 \overset{\alpha|0}{\rightarrow} q_2$. To all remaining transitions, i.e., those not belonging to

$$\{ (p_0,\alpha,p_1), (p_1,\beta,q_1), (p_0,\beta,p_2), (p_2,\alpha,q_2) \} \cup (Q_1 \times \Gamma \times Q_1) \cup (Q_2 \times \Gamma \times Q_2),$$

Figure 1. A weighted word automaton over $\mathcal{R}_{1/16}$ and $\Sigma = \{ \alpha, \beta \}$. 

\[ \begin{array}{c}
q_0 \quad \alpha|1/2, \beta|1/3 \quad \alpha|1/4, \beta|1 \quad 1/2 \\
\quad \quad \alpha|1/6 \quad \beta|1/2 \\
q_1 \quad \alpha|1/2 \quad \beta|2/3 \\
1 \quad q_2 \quad \alpha|3/4 \quad q_3 \\
\end{array} \]
we assign weight $\infty$. Moreover, state $p_0$ gets initial weight 0 whereas all other states get $\infty$. Finally, the states from $Q_1 \cup Q_2$ keep their final weights and the three new states get final weight $\infty$.

Concerning the behavior of $A$ we obtain

$$(\|A\|_w, \alpha \beta u) = (\|A_1\|_w, u) \quad \text{and} \quad (\|A\|_w, \beta \alpha u) = (\|A_2\|_w, u)$$

for any $u \in \Gamma^*$, whereas for all other words $w \in \Sigma^* \setminus \{\alpha \beta, \beta \alpha\} \Gamma^*$ we have $(\|A\|_w, w) = \infty$. Thus, the behavior of $A$ is trace closed w.r.t. $I = \{(\alpha, \beta), (\beta, \alpha)\}$ if and only if $\|A_1\|_w = \|A_2\|_w$ holds true. $\Box$

Despite this undecidability result, for weighted automata over semirings the situation was not lost. The $I$-diamond property turned out to be a sufficient condition for a trace closed behavior. A weighted word automaton $A = (Q, \text{in}, \text{wt}, \text{out})$ has the (weak) $I$-diamond property if for all $(\sigma, \tau) \in I$ and $p, r \in Q$ the following equation holds true:

$$\sum_{q \in Q} \text{wt}(p, \sigma, q) \cdot \text{wt}(q, \tau, r) = \sum_{q \in Q} \text{wt}(p, \tau, q) \cdot \text{wt}(q, \sigma, r).$$

However, proving that the $I$-diamond property implies a trace closed behavior depends on the distributivity of the semiring:

**Example 3.6.** We consider the non-distributive bounded lattice $M_3$ which is shown in Figure 3 and the independence alphabet $(\Sigma, I)$ with $\Sigma = \{\sigma, \tau\}$ and $I = \{(\sigma, \tau), (\tau, \sigma)\}$. It is easy to check that the weighted word automaton $A$ over $M_3$ and $\Sigma$ which is depicted in Figure 4 has the $I$-diamond property. However, its behavior is not trace closed since we have

$$(\|A\|_w, \sigma \tau) = a \land 1 = a \quad \text{and} \quad (\|A\|_w, \tau \sigma) = (a \land b) \lor (a \land c) = 0.$$

Although the $I$-diamond property does not imply a trace closed behavior for arbitrary strong bimonoids, we can give a syntactical property of weighted automata which emerges to be sufficient for trace closedness:
Definition 3.7. A weighted word automaton \( A = (Q, \text{in}, \text{wt}, \text{out}) \) has the strong \( I \)-diamond property if for all \( (\sigma, \tau) \in I \) and \( p, r \in Q \) there exists a bijection \( f = f_{p, r}^{\sigma, \tau} : Q \to Q \) such that for any \( q \in Q \) the following equation holds true:

\[
\text{wt}(p, \sigma, q) \cdot \text{wt}(q, \tau, r) = \text{wt}(p, \tau, f(q)) \cdot \text{wt}(f(q), \sigma, r).
\]

Clearly, an automaton with the strong \( I \)-diamond property also has the (weak) \( I \)-diamond property. Moreover, this property is decidable as soon as we can effectively compute and test for equality in \( A \).

Example 3.8. The weighted word automaton \( A \) from Example 3.1 has the strong \( I \)-diamond property w.r.t. \( I = \{ (\alpha, \beta), (\beta, \alpha) \} \). For \( f_{q_0, q_4}^{\alpha, \beta} \) and \( f_{q_0, q_4}^{\beta, \alpha} \) we can choose the bijection

\[
\{ q_0 \mapsto q_0, q_1 \mapsto q_2, q_2 \mapsto q_3, q_3 \mapsto q_1, q_4 \mapsto q_4 \}
\]

and its inverse. For all other mappings we can take the identity map.

Lemma 3.9. If a weighted word automaton has the strong \( I \)-diamond property, then its behavior is trace closed w.r.t. \( I \).

Proof:
Let \( A = (Q, \text{in}, \text{wt}, \text{out}) \) be a weighted word automaton having the strong \( I \)-diamond property. We have to show that all \( u, v \in \Sigma^* \) with \( u \sim v \) satisfy \((\|A\|_w, u) = (\|A\|_w, v)\). Due to the definition of \( \sim \) it suffices to consider \( u = x\sigma\tau y \) and \( v = x\tau\sigma y \) where \( x, y \in \Sigma^* \) and \( (\sigma, \tau) \in I \). We put \( k = |x\sigma| \) and \( n = |u| \). Then the mapping \( f : Q^{n+1} \to Q^{n+1} \) replacing \( q_k \) in \( (q_0, \ldots, q_n) \) by \( f_{q_0, \ldots, q_k+1}^{\sigma, \tau} (q_k) \) is a bijection such that in the sum defining the behavior of \( A \), cf. equation (1), the summand for \( (q_0, \ldots, q_n) \) and \( u \) equals the summand for \( f(q_0, \ldots, q_n) \) and \( v \). Thus, \((\|A\|_w, u) = (\|A\|_w, v)\). \( \square \)

Using the syntactic and decidable strong \( I \)-diamond property, we can define our model for weighted trace automata:

Definition 3.10. A weighted trace automaton over \( A \) and \( M \) is a weighted word automaton \( A \) over \( A \) and \( \Sigma \) having the strong \( I \)-diamond property. Its (trace) behavior is the trace series \( \|A\|_t \in A \langle \langle M \rangle \rangle \) defined by

\[
(\|A\|_t, [w]) = (\|A\|_w, w).
\]

A trace series \( S \in A \langle \langle M \rangle \rangle \) is called recognizable if there exists a weighted trace automaton such that \( S = \|A\|_t \).
Remark 3.11. Droste and Gastin [7] defined recognizability for trace series over semirings by weighted word automata having the (weak) I-diamond property. However, from Theorem 4.1 in [17] we can conclude that for commutative semirings both notions of recognizability coincide.

Remark 3.12. Recall that we can consider every word series in $A\langle\Sigma^*\rangle$ as a trace series in $A\langle M(\Sigma,\emptyset)\rangle$. Hence, there are seemingly two notions of recognizability for word series. But since every weighted word automaton has the strong $\emptyset$-diamond property, they coincide.

3.2. Rational expressions

The goal of this paper is to describe the behavior of weighted trace automata using weighted rational expressions. For this, we introduce some notation. For a trace series $S \in A\langle M\rangle$ we call the set

$$\text{supp}(S) = \{ t \in M \mid (S,t) \neq 0 \}$$

the support of $S$. A polynomial is a trace series with finite support. For $S, T \in A\langle M\rangle$ and $a \in A$ we define new trace series $a \cdot S, S + T, S \cdot T \in A\langle M\rangle$ called exterior product, sum, and Cauchy product by letting

$$(a \cdot S,t) = a \cdot (S,t), \quad (S + T,t) = (S,t) + (T,t), \quad \text{and} \quad (S \cdot T,t) = \sum_{t = uv} (S,u) \cdot (T,v).$$

Since the Cauchy product is associative precisely if $A$ is distributive, powers of $S$ need to be considered explicitly. For $n \in \mathbb{N}$ we define $S^n \in A\langle M\rangle$ by

$$(S^n,t) = \sum_{t = u_1 \cdots u_n} (S,u_1) \cdots (S,u_n).$$

If $S$ is proper, i.e. $(S,1_M) = 0$, we define the iteration $S^* \in A\langle M\rangle$ of $S$ by

$$(S^*,t) = \sum_{0 \leq n \leq |t|} (S^n,t),$$

where $|t|$ denotes the length of any representative of $t$. A trace series $S \in A\langle M\rangle$ is called rational if it can be constructed from the polynomials using sum, Cauchy product, and iteration of proper series. For the case of word series over semirings, Schützenberger's theorem states the equivalence of recognizability and rationality:

Theorem 3.13. (Schützenberger [21])

Let $K$ be a semiring and $S \in K\langle\Sigma^*\rangle$ a word series. Then $S$ is recognizable if and only if $S$ is rational.

As observed in [7], this theorem cannot be generalized directly to trace series due to problems concerning connectedness. A trace series $S \in A\langle M\rangle$ is called connected if $\text{supp}(S)$ is connected, and mono-alphabetic if $\text{alph}(t) = \text{alph}(t')$ for all $t, t' \in \text{supp}(S)$. We call $S$ c-rational if it can be constructed from the polynomials using sum, Cauchy product, and iteration of proper, connected series. If the iteration is further restricted to proper, mono-alphabetic, connected series, we call $S$ mc-rational.
Theorem 3.14. (Droste and Gastin [7])
Let $K$ be a semiring and $S \in K \langle\langle M \rangle\rangle$ a trace series.

1. If $S$ is recognizable, then $S$ is mc-rational.

2a. If $K$ is commutative and $S$ is mc-rational, then $S$ is recognizable.

2b. If $K$ is idempotent and commutative and $S$ is c-rational, then $S$ is recognizable.

Schützenberger’s theorem was also extended to bi-locally finite strong bimonoids:

Theorem 3.15. (Droste and Vogler [10])
Let $A$ be a bi-locally finite strong bimonoid and $S \in A \langle\langle \Sigma^* \rangle\rangle$ a word series. Then $S$ is recognizable if and only if $S$ is rational.

The main result of this paper jointly extends the three previous theorems. Unlike Theorem 3.14, we do not assume distributivity but bi-local finiteness. Moreover, we can eliminate commutativity from the premises of statement (2a).

Theorem 3.16. Let $A$ be a bi-locally finite strong bimonoid and $S \in A \langle\langle M \rangle\rangle$ a trace series.

1. If $S$ is recognizable, then $S$ is mc-rational.

2a. If $S$ is mc-rational, then $S$ is recognizable.

2b. If $A$ is idempotent and commutative and $S$ is c-rational, then $S$ is recognizable.

Since all bounded lattices are commutative, idempotent, and bi-locally finite strong bimonoids, we obtain the following corollary:

Corollary 3.17. Let $L$ be a bounded lattice and $S \in L \langle\langle M \rangle\rangle$ a trace series. The following are equivalent:

(i) $S$ is recognizable,

(ii) $S$ is mc-rational,

(iii) $S$ is c-rational.

In order to prove the main theorem we need the concept of recognizable step series.

3.3. Recognizable step series

For a language $L \subseteq M$ we define its characteristic series $\chi_{L,A} \in A \langle\langle M \rangle\rangle$ by

\[
(\chi_{L,A}, t) = \begin{cases} 
1 & \text{if } t \in L, \\
0 & \text{if } t \notin L.
\end{cases}
\]

A trace series $S \in A \langle\langle M \rangle\rangle$ is called recognizable step series if there are $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$ and recognizable trace languages $L_1, \ldots, L_n \subseteq M$ such that

\[
S = \sum_{i=1}^{n} a_i \cdot \chi_{L_i,A}.
\]
Note that the word “recognizable” in this notion does not refer to the recognizability of the whole series but to the recognizability of the steps. However, in Proposition 3.19 we will show that recognizable step series are indeed recognizable in the sense of Definition 3.10.

In the following, we use the characterization below without explicitly mentioning it.

**Lemma 3.18.** A trace series \( S \in A\langle\langle M \rangle\rangle \) is a recognizable step series if and only if \( \text{im}(S) \) is finite and \( S^{-1}(a) \) is recognizable for each \( a \in \text{im}(S) \).

**Proof:**
The if-part follows from \( S = \sum_{a \in \text{im}(S)} a \cdot \chi_{S^{-1}(a),A} \).

For the converse direction, consider \( a_1, \ldots, a_n \in A \) and recognizable trace languages \( L_1, \ldots, L_n \subseteq M \) such that \( S = \sum_{i=1}^n a_i \cdot \chi_{L_i,A} \). Then

\[
\text{im}(S) \subseteq \left\{ \sum_{j \in J} a_j \ \Big| \ J \subseteq \{1, \ldots, n\} \right\},
\]

and hence \( \text{im}(S) \) has at most \( 2^n \) elements. Furthermore, for all \( a \in A \) we have

\[
S^{-1}(a) = \bigcup \left\{ \bigcap_{j \in J} L_j \setminus \bigcup_{j \notin J} L_j \ \Big| \ J \subseteq \{1, \ldots, n\}, \sum_{j \in J} a_j = a \right\}.
\]

Since the class of recognizable trace languages is closed under boolean operations, \( S^{-1}(a) \) is recognizable. \( \square \)

The following proposition shows that recognizable step series are not only step series whose steps are recognizable but are recognizable themselves.

**Proposition 3.19.** Every recognizable step series \( S \in A\langle\langle M \rangle\rangle \) is recognizable.

**Proof:**
Let \( S = \sum_{i=1}^n a_i \cdot \chi_{L_i,A} \). For each \( i = 1, \ldots, n \) let \( h_i : M \to S_i \) be a monoid morphism into a finite monoid \( S_i \) and \( F_i \subseteq S_i \) be such that \( L_i = h_i^{-1}(F_i) \). Moreover, let \( Q = \bigcup_{1 \leq i \leq n} \{i\} \times S_i \) be the disjoint union of the sets \( S_i \). We define a weighted word automaton \( A = (Q, \text{in}, \text{wt}, \text{out}) \) by letting

\[
\text{in}((i,p)) = \begin{cases} 1 & \text{if } p = 1_{S_i}, \\ 0 & \text{otherwise}, \end{cases} \quad \text{out}((j,q)) = \begin{cases} a_j & \text{if } q \in F_j, \\ 0 & \text{otherwise}, \end{cases}
\]

and

\[
\text{wt}((i,p),\sigma,(j,q)) = \begin{cases} 1 & \text{if } i = j \text{ and } q = p \cdot h_i([\sigma]), \\ 0 & \text{otherwise}. \end{cases}
\]
First, we show that \( A \) satisfies the strong \( I \)-diamond property. Therefore, consider \((i, p), (k, r) \in Q\) and \((\sigma, \tau) \in I\). If \(i \neq k\) or \(i = k\) but \( r \neq p \cdot h_i([\sigma \tau]) \), then the products
\[
\text{wt}((i, p), \sigma, (j, q)) \cdot \text{wt}((j, q), \tau, (k, r)) \quad \text{and} \quad \text{wt}((i, p), \tau, (j, q)) \cdot \text{wt}((j, q), \sigma, (k, r))
\]
are both equal to 0 for any \((j, q) \in Q\). Thus, we can choose the identity on \( Q \) as bijection \( f_{p,\tau}^{\sigma,\tau} \). Now, let us assume \(i = k\) and \(r = p \cdot h_i([\sigma \tau])\). The first product in (2) is different from 0 precisely if \(i = j\) and \( q = p \cdot h_i([\sigma]) \), and then its value is 1. Analogously, the second product in (2) equals 1 if \(i = j\) and \( q = p \cdot h_i([\tau]) \), and 0 in all other cases. Hence, we can choose \( f_{p,\tau}^{\sigma,\tau} \) to be the mapping which swaps \( p \cdot h_i([\sigma]) \) and \( p \cdot h_i([\tau])\). This shows that \( A \) has the strong \( I \)-diamond property.

Finally, we prove that \( A \) has the desired behavior, i.e., \( \|A\|_A = S \). For this, let \( t = [\sigma_1 \ldots \sigma_m] \in \mathbb{M} \) and \((i_0, q_0), \ldots, (i_m, q_m) \in Q\). The product
\[
\text{im}((i_0, q_0)) \cdot \text{wt}((i_0, q_0), \sigma_1, (i_1, q_1)) \cdots \text{wt}((i_{m-1}, q_{m-1}), \sigma_m, (i_m, q_m)) \cdot \text{out}((i_m, q_m))
\]
can only differ from 0, if all the \( i_\ell \) simultaneously equal some \( i \), \( q_0 = 1_{S_i} \), and \( q_\ell = q_{\ell-1} \cdot h_i([\sigma_\ell]) \) for each \( \ell = 1, \ldots, m \). In this situation we have \( q_m = h_i(t) \) and the product in (3) evaluates to \( a_i \) if \( h_i(t) \in F_i \), i.e., \( t \in L_i \), and 0 otherwise. Finally,
\[
(\|A\|_A, t) = (\|A\|_m, \sigma_1 \ldots \sigma_m) = \sum_{i=1}^n a_i \cdot (\chi_{L_i \setminus A}, t) = (S, t)
\]
proves the claim. \(\square\)

**Corollary 3.20.** Let \( L \subseteq \mathbb{M} \) be a recognizable trace language. Then \( \chi_{L \setminus A} \) is recognizable.

The converse of Proposition 3.19 holds under the additional assumption of bi-local finiteness. In order to prove this we need another notion. For a trace series \( S \in A\langle\mathbb{M}\rangle \) we define a word series \( \varphi^{-1}(S) \in A\langle\Sigma^+\rangle \) by letting
\[
(\varphi^{-1}(S), w) = (S, [w]).
\]

**Lemma 3.21.** Let \( S \in A\langle\mathbb{M}\rangle \) be a trace series. Then \( S \) is a recognizable step series if and only if \( \varphi^{-1}(S) \) is a recognizable step series.

**Proof:**
The claim follows directly from \( \text{im}(S) = \text{im}(\varphi^{-1}(S)) \), \( \varphi^{-1}(S^{-1}(a)) = (\varphi^{-1}(S))^{-1}(a) \) for all \( a \in A \), and Proposition 2.1. \(\square\)

The following theorem generalizes a result of Droste, Stüber, and Vogler [9] which states the word case:

**Proposition 3.22.** If \( A \) is bi-locally finite, then every recognizable trace series \( S \in A\langle\mathbb{M}\rangle \) is a recognizable step series.

**Proof:**
Let \( A \) be a weighted trace automaton with \( \|A\|_A = S \). If we consider the underlying weighted word automaton, we obtain \( \|A\|_w = \varphi^{-1}(S) \), and hence \( \varphi^{-1}(S) \) is recognizable. By the word case of Proposition 3.22 due to [9], \( \varphi^{-1}(S) \) is a recognizable step series, and by Lemma 3.21 \( S \) is a recognizable step series as well. \(\square\)

This result turns out to be very important for the proof of Theorem 3.16, since it allows us to use the properties of recognizable step series whenever we consider recognizable trace series.
4. Recognizable trace series are mc-rational

This section is devoted to the proof of the following proposition which implies Theorem 3.16 (1):

**Proposition 4.1.** Every recognizable step series \( S \in A(\langle M \rangle) \) is mc-rational.

Since mc-rationality implies c-rationality by definition, this shows as well that recognizable step series are c-rational. The main idea behind the proof is captured by the following lemma:

**Lemma 4.2.** Let \( L \subseteq M \) be a recognizable trace language. Then \( \chi_{L,A} \) is mc-rational.

In the following we use the symbol \( \mathbb{N} \) also to denote the semiring \((\mathbb{N}, +, \cdot, 0, 1)\) of the natural numbers, where + and \( \cdot \) are the usual addition and multiplication.

**Proof:**

By Corollary 3.20 and Theorem 3.14 the characteristic series \( \chi_{L,N} \) is mc-rational. Hence, it suffices to show that for all trace languages \( L \subseteq M \) the series \( \chi_{L,A} \) is mc-rational if \( \chi_{L,N} \) is mc-rational. We prove this claim by induction on the mc-rational construction of \( \chi_{L,N} \).

First, if \( \chi_{L,N} \) is a polynomial, then \( L \) is finite and \( \chi_{L,A} \) is a polynomial, too. Second, we assume \( \chi_{L,N} = S_1 + S_2 \) for some mc-rational \( S_1, S_2 \in \mathbb{N}(\langle M \rangle) \). Since \( \text{im}(\chi_{L,N}) \subseteq \{ 0, 1 \} \), there are disjoint \( L_1, L_2 \subseteq M \) such that \( S_i = \chi_{L_i,N} \) for \( i = 1, 2 \). We obtain \( \chi_{L,A} = \chi_{L_1,A} + \chi_{L_2,A} \). By induction, \( \chi_{L_1,A} \) and \( \chi_{L_2,A} \) are mc-rational and so is \( \chi_{L,A} \).

Third, we consider \( \chi_{L,N} = S_1 \cdot S_2 \) with mc-rational series \( S_1, S_2 \in \mathbb{N}(\langle M \rangle) \). If there is a \( t \in M \) with \( (S_1, t) > 1 \), then \( \text{supp}(S_2) = \emptyset \). But this would imply \( L = \emptyset \) and we were done. Hence, we can assume there is some \( L_1 \subseteq M \) such that \( S_1 = \chi_{L_1,N} \). Similarly, we obtain \( S_2 = \chi_{L_2,N} \) for some \( L_2 \subseteq M \). For every \( t \in L \) there is at most one pair \((t_1, t_2)\) with \( t_1 \in L_1, t_2 \in L_2 \), and \( t = t_1t_2 \), since otherwise we would have \( (\chi_{L,N}, t) > 1 \). Hence, we conclude \( \chi_{L,A} = \chi_{L_1,A} \cdot \chi_{L_2,A} \). By induction, \( \chi_{L_1,A} \) and \( \chi_{L_2,A} \) are mc-rational and so is \( \chi_{L,A} \).

Finally, assume \( \chi_{L,N} = S_1^n \) for some mc-rational, proper, mono-alphabetic, and connected \( S_1 \in \mathbb{N}(\langle M \rangle) \). Since \( (S_1^n, t) \geq (S_1, t) \) for any \( t \in M \) there is some \( L_1 \subseteq M \) with \( S_1 = \chi_{L_1,N} \). For each \( t \in M \) there are at most one \( n \in \mathbb{N} \) and one tuple \((t_1, \ldots, t_n)\) with \( t_1, \ldots, t_n \in L_1 \) and \( t = t_1 \cdots t_n \), since otherwise we would have \( (\chi_{L,N}, t) > 1 \). Thus, we obtain \( \chi_{L,A} = (\chi_{L_1,A})^n \). Moreover, \( \text{supp}(\chi_{L_1,A}) = \text{supp}(\chi_{L_1,N}) \) implies that \( \chi_{L_1,A} \) is proper, mono-alphabetic, and connected as well. By induction, \( \chi_{L_1,A} \) is mc-rational and so is \( \chi_{L,A} \).

Using this lemma, we can give the missing proof of Proposition 4.1:

**Proof:**

Consider \( a_1, \ldots, a_n \in A \) and recognizable trace languages \( L_1, \ldots, L_n \subseteq M \) such that \( S = \sum_{i=1}^n a_i \cdot \chi_{L_i,A} \). By Lemma 4.2, the series \( \chi_{L_i,A} \) is mc-rational for each \( i = 1, \ldots, n \). Since

\[
a_i \cdot \chi_{L_i,A} = (a_i \cdot \chi_{\{1\}_M,A}) \cdot \chi_{L_i,A}
\]

and since \( a_i \cdot \chi_{\{1\}_M,A} \) is a polynomial, \( S \) is a finite sum of mc-rational series, and hence itself mc-rational. \( \square \)
5. mc-rational trace series are recognizable

This section is devoted to the proof of the following proposition which implies Theorem 3.16 (2a):

**Proposition 5.1.** If $A$ is bi-locally finite, then every mc-rational trace series $S \in A \langle \langle M \rangle \rangle$ is a recognizable step series.

We prove this result by induction on the mc-rational construction of $S$. For polynomials $S \in A \langle \langle M \rangle \rangle$ the claim follows from the finiteness of $\text{supp}(S)$ and the sum

$$S = \sum_{t \in \text{supp}(S)} (S, t) \cdot \chi_{\{t\},A}.$$

Closure of the class of recognizable step series under sum is obvious. In order to prove closure under Cauchy product, we need the following proposition which is part of Theorem 4.1 in [17]:

**Proposition 5.2. (Kuske [17])**
Let $K$ be a commutative semiring. A trace series $S \in K \langle \langle M \rangle \rangle$ is recognizable if and only if $\varphi^{-1}(S)$ is recognizable.

The word case of the following lemma is contained in Section 3.2 of [1]:

**Lemma 5.3.** Let $S \in \mathbb{N} \langle \langle M \rangle \rangle$ be a recognizable trace series. Then for all $k, \ell \in \mathbb{N}$ the trace language $S^{-1}(k + \ell \cdot \mathbb{N})$ is recognizable.

**Proof:**
Due to Proposition 5.2 the word series $\varphi^{-1}(S)$ is recognizable. Hence, the word case implies that $(\varphi^{-1}(S))^{-1}(k + \ell \cdot \mathbb{N})$ is a recognizable word language. Finally, the claim follows from

$$\varphi^{-1}(S^{-1}(k + \ell \cdot \mathbb{N})) = (\varphi^{-1}(S))^{-1}(k + \ell \cdot \mathbb{N})$$

and Proposition 2.1. \qed

Now, we are prepared to prove the closure under Cauchy product:

**Proposition 5.4.** Let $A$ be additively locally finite. If $S, T \in A \langle \langle M \rangle \rangle$ are recognizable step series, then $S \cdot T$ is also a recognizable step series.

**Proof:**
From the definition

$$(S \cdot T, t) = \sum_{u \in uv} (S, u) \cdot (T, v)$$

we observe that all possible summands are of the form $a \cdot b$ with $a \in \text{im}(S)$ and $b \in \text{im}(T)$. Thus, the main idea is to count how often each product $a \cdot b$ occurs. This is accomplished by the mapping $C_{a,b} : M \rightarrow \mathbb{N}$ defined as

$$C_{a,b}(t) = \text{card} \{ (u, v) \mid u \in S^{-1}(a), v \in T^{-1}(b), t = uv \}.$$
Moreover, let \( \eta_{a,b} : (\mathbb{N}, +) \to (A, +) \) be the unique monoid morphism satisfying \( \eta_{a,b}(1) = a \cdot b \). Using the composition \( U_{a,b} = \eta_{a,b} \circ C_{a,b} : \mathcal{M} \to A \), we can express \( S \cdot T \) as

\[
S \cdot T = \sum_{a \in \text{im}(S)} U_{a,b} \cdot b \in \text{im}(T)
\]

Since \( S \) and \( T \) are recognizable step series their images are finite. Thus, it suffices to show that \( U_{a,b} \in A \langle \langle \mathcal{M} \rangle \rangle \) is a recognizable step series for all \( a, b \in A \). Clearly, \( \text{im}(U_{a,b}) \) is finite since \( \text{im}(U_{a,b}) \subseteq \text{im}(\eta_{a,b}) \) and \((A, +) \) is locally finite. Now, consider \( c \in \text{im}(U_{a,b}) \). Since \( \eta_{a,b} \) is a monoid morphism from \((\mathbb{N}, +)\) to \( A \) with finite image and \( c \in \text{im}(\eta_{a,b}) \), there are \( k, \ell \in \mathbb{N} \) such that \( \eta_{a,b}^{-1}(c) = k + \ell \cdot \mathbb{N} \). Thus,

\[
U_{a,b}^{-1}(c) = C_{a,b}^{-1}(k + \ell \cdot \mathbb{N}) .
\]

Considering the series \( \chi_{S^{-1}(a),\mathbb{N}}, \chi_{T^{-1}(b),\mathbb{N}} \), the definition of \( C_{a,b} \) immediately implies

\[
C_{a,b} = \chi_{S^{-1}(a),\mathbb{N}} \cdot \chi_{T^{-1}(b),\mathbb{N}} .
\]

Since \( S \) and \( T \) are recognizable step series, \( S^{-1}(a) \) and \( T^{-1}(b) \) are recognizable trace languages. Thus, \( \chi_{S^{-1}(a),\mathbb{N}}, \chi_{T^{-1}(b),\mathbb{N}} \) are recognizable trace series. By Theorem 3.14 their Cauchy product \( C_{a,b} \) is also recognizable, and by Lemma 5.3, \( U_{a,b}^{-1}(c) \) is a recognizable trace language.

\[\square\]

Our next goal is to prove the closure under iteration of proper, mono-alphabetic, connected series. This uses a similar but more involved technique which makes some preliminary considerations inevitable.

Let \( K \) be a commutative semiring, \( \mathcal{M}_1 = \mathcal{M}(\Sigma_1, I_1) \) and \( \mathcal{M}_2 = \mathcal{M}(\Sigma_2, I_2) \) be two trace monoids, and \( h : \Sigma_1 \to K \langle \langle \mathcal{M}_2 \rangle \rangle \) be a mapping such that for all \( (\sigma, \sigma') \in I_1 \) we have

\[
h(\sigma) \cdot h(\sigma') = h(\sigma') \cdot h(\sigma).
\]

This condition implies that we can uniquely extend \( h \) to a monoid morphism \( \bar{h} : \mathcal{M}_1 \to (K \langle \langle \mathcal{M}_2 \rangle \rangle, \cdot) \). Moreover, suppose that \( h(\sigma) \) is proper for each \( \sigma \in \Sigma_1 \). Then, for \( s \in \mathcal{M}_1 \) and \( t \in \mathcal{M}_2 \) we obtain \( (\bar{h}(s), t) = 0 \) whenever \( |s| > |t| \). This allows us to define for every \( S \in K \langle \langle \mathcal{M}_1 \rangle \rangle \) a trace series \( \bar{h}(S) \in K \langle \langle \mathcal{M}_2 \rangle \rangle \) as

\[
\bar{h}(S) = \sum_{s \in M_1} (S, s) \cdot \bar{h}(s) .
\] (4)

Notice that for any \( t \in \mathcal{M}_2 \) the sum \((\bar{h}(S), t)\) is finite since only those \( s \in \mathcal{M}_1 \) with \( |s| \leq |t| \) contribute to it. Using exactly the same techniques as in Section 1.4 of [1], we can show that this determines a semiring morphism \( h : K \langle \langle \mathcal{M}_1 \rangle \rangle \to K \langle \langle \mathcal{M}_2 \rangle \rangle \) which preserves properness and is compatible with iteration. We are interested in sufficient conditions under which \( \bar{h} \) preserves recognizability. By Theorem 3.14, we can equivalently consider mc-rationality instead of recognizability. First, assume that \( h(\sigma) \) is mc-rational for each \( \sigma \in \Sigma_1 \). Then for any polynomial \( S \in K \langle \langle \mathcal{M}_1 \rangle \rangle \) the series \( \bar{h}(S) \) is a finite sum of finite Cauchy products of mc-rational series, and hence itself mc-rational. Since \( \bar{h} \) is compatible with sum, Cauchy product, and iteration, it remains to investigate conditions under which \( \bar{h} \) preserves mono-alphabetic connectedness. Therefore, let \( S \in K \langle \langle \mathcal{M}_1 \rangle \rangle \) be a proper, mono-alphabetic, connected trace series, and let \( X \subseteq \Sigma_1 \) be such that \( \text{alph}(s') = X \) for any \( s' \in \text{supp}(S) \). From (4) we conclude, that for each
$t \in \text{supp}(\tilde{h}(S))$ there is an $s \in \text{supp}(S)$ such that $t \in \text{supp}(\tilde{h}(s))$. We further assume that for any $\sigma \in \Sigma_1$ there is a set $Y_\sigma \subseteq \Sigma_2$ satisfying $\text{alph}(t') = Y_\sigma$ for all $t' \in \text{supp}(h(\sigma))$. Then we can conclude $\text{alph}(t) = \bigcup_{\sigma \in X} Y_\sigma$. It follows that $\tilde{h}(S)$ is always mono-alphabetic, and connected if $\bigcup_{\sigma \in X} Y_\sigma$ is connected. The following technical lemma summarizes these observations:

**Lemma 5.5.** Let $K$ be a commutative semiring, $M_1 = M(\Sigma_1, I_1)$ and $M_2 = M(\Sigma_2, I_2)$ be two trace monoids, and $h: \Sigma_1 \to K\langle\langle M_2\rangle\rangle$ be a mapping satisfying the following conditions:

(i) $h(\sigma)$ is proper and recognizable for any $\sigma \in \Sigma_1$,

(ii) $h(\sigma) \cdot h(\sigma') = h(\sigma') \cdot h(\sigma)$ for all $(\sigma, \sigma') \in I_1$,

(iii) for each $\sigma \in \Sigma_1$ there is a set $Y_\sigma \subseteq \Sigma_2$ such that $\text{alph}(t) = Y_\sigma$ for all $t \in \text{supp}(h(\sigma))$,

(iv) the subalphabet $\bigcup_{\sigma \in X} Y_\sigma \subseteq \Sigma_2$ is connected whenever $X \subseteq \Sigma_1$ is connected.

Then the mapping $\tilde{h}: K\langle\langle M_1\rangle\rangle \to K\langle\langle M_2\rangle\rangle$ defined as in (4) preserves recognizability.

Using these considerations we can prove the closure under iteration of proper, mono-alphabetic, connected series.

**Proposition 5.6.** Let $A$ be bi-locally finite. If $S \in A\langle\langle M\rangle\rangle$ is a proper, mono-alphabetic, connected recognizable step series, then $S^*$ is also a recognizable step series.

**Proof:**
Recall that

$$(S^*, t) = \sum_{0 \leq n \leq |t| \atop t = u_1 \cdot \ldots \cdot u_n} (S, u_1) \cdots (S, u_n).$$

Again, we are interested in how often each addend $a \in A$ occurs. However, we need to take a slightly more general approach than in the proof of Proposition 5.4. Thus, we put $B = \text{im}(S) \setminus \{0\}$ and define for any $L \subseteq B^*$ and $t \in M$ a set $D_L(t)$ as

$$D_L(t) = \{ (u_1, \ldots, u_n) \mid 0 \leq n \leq |t|, u_i \in \text{supp}(S), t = u_1 \cdot \ldots \cdot u_n, ((S, u_1), \ldots, (S, u_n)) \in L \}.$$

Since this set is finite, we are able to define a mapping $C_L: M \to N$ by

$$C_L(t) = \text{card } D_L(t).$$

Let $\psi: B^* \to (A, \cdot)$ denote the unique extension of $B \to A$ to a monoid morphism. For any $a \in A$ we obtain

$$C_{\psi^{-1}(a)}(t) = \text{card } \{ (u_1, \ldots, u_n) \mid 0 \leq n \leq |t|, u_i \in \text{supp}(S), t = u_1 \cdot \ldots \cdot u_n, (S, u_1) \cdots (S, u_n) = a \}.$$

Using the unique monoid morphism $\eta_a: (N, +) \to (A, +)$ with $\eta_a(1) = a$, we can express $S^*$ as

$$S^* = \sum_{a \in \text{im}(\psi)} \eta_a \circ C_{\psi^{-1}(a)}.$$
This sum is finite, since \((A, \cdot)\) is locally finite. Thus, using the same techniques as in the proof of Proposition 5.4 it suffices to show that \(C_{\psi^{-1}(a)} \in \mathbb{N}\langle\langle \mathbb{M} \rangle\rangle\) is recognizable for any \(a \in A\).

First, we show that the mapping \(h: B \to \mathbb{N}\langle\langle \mathbb{M} \rangle\rangle, b \mapsto \chi_{S^{-1}(b), N}\) satisfies the four conditions of Lemma 5.5 for \(\mathbb{M}_1 = \mathbb{M}(\emptyset, B) = B^*\). Since \(S\) is a proper recognizable step series, \(h(b)\) is proper, and by Corollary 3.20, also recognizable. Due to the choice of \(\mathbb{M}_1\), the second condition is trivially satisfied. Let \(X \subseteq \Sigma\) be a connected set such that \(\text{alph}(t) = X\) for any \(t \in \text{supp}(S)\). Then we can choose \(Y_b = X\) for any \(b \in B\) and the fourth condition is obviously met. Thus, the mapping \(\tilde{h}: \mathbb{N}\langle\langle B^* \rangle\rangle \to \mathbb{N}\langle\langle \mathbb{M} \rangle\rangle\), defined as in (4), preserves recognizability.

Next, we show that for all \(L \subseteq B^*\) we have \(\tilde{h}(\chi_{L,N}) = C_L\). Therefore, let \(\tilde{h}: B^* \to \langle\langle \mathbb{N}\langle\langle \mathbb{M} \rangle\rangle, \cdot\rangle\rangle\) be the unique extension of \(h\) to a monoid morphism. For \(w = b_1 \ldots b_n \in B^*\) we obtain

\[
(\tilde{h}(w), t) = \sum_{t=u_1 \ldots u_n} (\chi_{S^{-1}(b_1), N}, u_1) \cdots (\chi_{S^{-1}(b_n), N}, u_n)
\]

\[
= \text{card} \left\{ (u_1, \ldots, u_n) \mid u_i \in \text{supp}(S), t = u_1 \cdots u_n, (S, u_i) = b_i \right\}
\]

\[
= C_{\{w\}}(t).
\]

Moreover, for all languages \(L_1, L_2 \subseteq B^*\) and traces \(t \in \mathbb{M}\) we can conclude from \((u_1, \ldots, u_n) \in D_{L_1}(t) \cap D_{L_2}(t)\) that \((S, u_1), \ldots, (S, u_n)\) \(\in L_1 \cap L_2\). Hence, if \(L_1\) and \(L_2\) are disjoint, then \(D_{L_1}(t)\) and \(D_{L_2}(t)\) are also disjoint. Since \(C_{\{w\}}(t) = (\tilde{h}(w), t) = 0\) whenever \(|w| > |t|\), it follows

\[
C_L(t) = \sum_{w \in L} C_{\{w\}}(t) = \sum_{w \in B^*} (\chi_{L,N}, w) \cdot (\tilde{h}(w), t) = (\tilde{h}(\chi_{L,N}), t).
\]

Finally, \(\psi^{-1}(a)\) is recognizable since \(\text{im}(\psi)\) is finite. By Corollary 3.20, \(\chi_{\psi^{-1}(a), N}\) is recognizable. Since \(\tilde{h}\) preserves recognizability, \(C_{\psi^{-1}(a)} = \tilde{h}(\chi_{\psi^{-1}(a), N})\) is also recognizable. As mentioned above, this shows the claim. \(\square\)

6. c-rational trace series are recognizable

This section is devoted to the proof of Theorem 3.16 (2b). Moreover, we give two counterexamples showing that we can neither eliminate idempotency nor commutativity from the assumptions.

6.1. The proof

First, we show the following proposition which implies Theorem 3.16 (2b):

Proposition 6.1. If \(A\) is idempotent, commutative, and bi-locally finite, then every c-rational trace series \(S \in A\langle\langle \mathbb{M} \rangle\rangle\) is a recognizable step series.

Again, we prove this result by induction on the c-rational construction of \(S\). It was shown in the last section that the class of recognizable step series contains the polynomials, and is closed under sum and Cauchy product. It remains to prove the closure under iteration of proper, connected series.

Proposition 6.2. Let \(A\) be idempotent, commutative, and bi-locally finite. If \(S \in A\langle\langle \mathbb{M} \rangle\rangle\) is a proper, connected recognizable step series, then \(S^*\) is also a recognizable step series.
Proof:
Since $A$ is idempotent, in contrast to the proof Proposition 5.6, we do not need to count occurrences of $a \in A$ in the sum defining $(S^*, t)$ for a given trace $t \in \mathbb{M}$. It only matters if $a$ occurs at least once. Therefore, we define a trace language $E_a \subseteq \mathbb{M}$ as

$$E_a = \{ t_1 \cdots t_n \mid n \in \mathbb{N}, t_1, \ldots, t_n \in \text{supp}(S), (S, t_1) \cdots (S, t_n) = a \}.$$  

Let $B = \text{im}(S) \setminus \{0\}$ and $C$ denote the submonoid of $(A, \cdot)$ generated by $B$. Then we obtain

$$S^* = \sum_{a \in C} a \cdot \chi_{E_a, A}.$$  

Since $C$ is finite, it suffices to show that $E_a$ is recognizable for any $a \in C$. Note that we cannot adopt directly the proof strategy for Proposition 5.6 due to problems with the connectedness. However, using an auxiliary construction we can circumvent these issues. Therefore, we define an independence alphabet $(\Gamma, H)$ by letting

$$\Gamma = \{ (b, X) \mid b \in B, X \subseteq \Sigma \text{ is non-empty and connected} \}$$

and

$$((b, X), (c, Y)) \in H \iff X \times Y \subseteq I \land (b, X) \neq (c, Y).$$

Since $A$ is commutative, we can extend the projection $\Gamma \to C, (b, X) \mapsto b$ to a monoid morphism $\psi: \mathbb{M}(\Gamma, H) \to C$ in a unique way. Moreover, for $t \in \text{supp}(S)$ we denote the pair $((S, t), \text{alph}(t))$ by $\gamma(t)$. Since $S$ is proper and connected, this defines a mapping $\gamma: \text{supp}(S) \to \Gamma$. For each $U \subseteq \mathbb{M}(\Gamma, H)$ we define a trace language $D(U) \subseteq \mathbb{M}$ as

$$D(U) = \{ t_1 \cdots t_n \mid n \in \mathbb{N}, t_1, \ldots, t_n \in \text{supp}(S), [\gamma(t_1) \cdots \gamma(t_n)] \in U \}.$$  

Then, for any $a \in C$ we obtain

$$E_a = D(\psi^{-1}(a)).$$

Now, we show that the mapping

$$h: \Gamma \to \mathbb{B}[[\mathbb{M}]], \beta \mapsto \chi_{\gamma^{-1}(\beta), \mathbb{B}},$$

where $\mathbb{B}$ denotes the boolean semiring, satisfies the four conditions of Lemma 5.5 for $\mathbb{M}_1 = \mathbb{M}(\Gamma, H)$. Clearly, $h(\beta)$ is proper since $S$ is proper. For $\beta = (b, X) \in \Gamma$ we have

$$\gamma^{-1}(\beta) = S^{-1}(b) \cap \{ t \in \mathbb{M} \mid \text{alph}(t) = X \}.$$  

It is easy to see that the latter set is recognizable and so is $\gamma^{-1}(\beta)$. By Corollary 3.20, $h(\beta)$ is recognizable as well. In order to prove that the second condition is met, consider $(\alpha, \beta) \in H, s \in \gamma^{-1}(\alpha)$, and $t \in \gamma^{-1}(\beta)$. By the definition of $H$ we have $\text{alph}(s) \times \text{alph}(t) \subseteq I$, and hence $st = ts \in \gamma^{-1}(\beta) \cdot \gamma^{-1}(\alpha)$. This implies $\gamma^{-1}(\alpha) \cdot \gamma^{-1}(\beta) \subseteq \gamma^{-1}(\beta) \cdot \gamma^{-1}(\alpha)$, and the converse inclusion follows analogously. Thus, $h(\alpha) \cdot h(\beta) = h(\beta) \cdot h(\beta)$.

Concerning the third condition, we can choose $Y_{(b, X)} = X$ for all $(b, X) \in \Gamma$. Finally, let $Z \subseteq \Gamma$ be connected. Then we have to show that $X = \bigcup_{(b, X) \in Z} X$ is connected. Indirectly, assume there is
a partition \( \mathcal{X} = A \cup B \) into non-empty sets such that \( A \times B \subseteq I \). Let \( C = \{(b, X) \in \mathbb{Z} \mid X \subseteq A\} \) and \( D = \{(b, X) \in \mathbb{Z} \mid X \subseteq B\} \). For each \( (b, X) \in \mathbb{Z} \) we obtain \( X = (X \cap A) \cup (X \cap B) \) and since \( A \) is connected, this implies either \( X \subseteq A \) or \( X \subseteq B \), i.e., \( \mathbb{Z} = C \cup D \) and this union is disjoint. Since \( I \) is irreflexive, \( A \) and \( B \) are disjoint, and hence \( C \) and \( D \) are both non-empty. For all \( (b, X) \in C \) and \( (c, Y) \in D \) we have \( X \times Y \subseteq A \times B \subseteq I \), i.e., \( ((b, X), (c, Y)) \in H \). Thus, \( C \times D \subseteq H \). But this contradicts the connectedness of \( H \), hence \( \mathcal{X} \) is connected. It follows that the mapping \( \tilde{h} : \mathbb{B} \langle \mathbb{M}(\Gamma, H) \rangle \rightarrow \mathbb{B} \langle \mathbb{M} \rangle \), defined as in (4), preserves recognizability.

Next, we show \( \text{supp}(\tilde{h}(\chi_{U,B})) = D(U) \) for all \( U \subseteq \mathbb{M}(\Gamma, H) \). Let \( h : \mathbb{M}(\Gamma, H) \rightarrow (\mathbb{B} \langle \mathbb{M} \rangle, \cdot) \) be the unique extension of \( h \) to a monoid morphism. It is easy to see that \( \text{supp}(\tilde{h}(u)) = D(\{u\}) \) for any \( u \in \mathbb{M}(\Gamma, H) \). Thus, for \( U \subseteq \mathbb{M}(\Gamma, H) \) we have

\[
\text{supp}(\tilde{h}(\chi_{U,B})) = \bigcup_{u \in U} \text{supp}((\chi_{U,B}, u) \cdot h(u)) = \bigcup_{u \in U} \text{supp}(h(u)) = \bigcup_{u \in U} D(\{u\}) = D(U).
\]

Putting the pieces together, we obtain

\[
E_a = D(\psi^{-1}(a)) = \text{supp}(\tilde{h}(\chi_{\psi^{-1}(a),B})) = \left(\tilde{h}(\chi_{\psi^{-1}(a),B})\right)^{-1}(1).
\]

Since \( C \) is finite, \( \psi^{-1}(a) \) is a recognizable trace language, hence \( \chi_{\psi^{-1}(a),B} \) is recognizable. As \( \tilde{h} \) preserves recognizability and by Proposition 3.22, \( \tilde{h}(\chi_{\psi^{-1}(a),B}) \) is a recognizable step series. Thus, \( E_a \) is recognizable and the claim follows.

\[ \square \]

6.2. The counterexamples

Finally, we give the two counterexamples showing that we can neither drop idempotence nor commutativity from the assumptions of Theorem 3.16 (2b) and Propositions 6.1 and 6.2. For both examples let \( \mathbb{M} \) be the trace monoid \( \mathbb{M} = \mathbb{M}(\Sigma, I) \) with \( \Sigma = \{\sigma, \tau\} \) and \( I = \{(\sigma, \tau), (\tau, \sigma)\} \).

Example 6.3. Let \( \mathbb{F}_2 \) be the field with two elements. Consider the series

\[
S = [\sigma] + [\tau] \in \mathbb{F}_2 \langle \mathbb{M} \rangle.
\]

Then \( S^* \) is obviously c-rational. Using the canonical identification \( \mathbb{M} \cong \mathbb{N} \times \mathbb{N} \), which maps \( t \in \mathbb{M} \) to \( (|t|_\sigma, |t|_\tau) \in \mathbb{N} \times \mathbb{N} \), we have

\[
(S^*, (m, n)) = (S^{m+n}, (m, n)) = \binom{m+n}{m} \mod 2.
\]

Thus, \( L = (S^*)^{-1}(0) \) is the set of all pairs \((m, n)\) such that \( \binom{m+n}{m} \) is even. By counting prime factors we obtain for all \( k, \ell \in \mathbb{N} \) that \( (2^k, 2^\ell) \in L \) holds true precisely if \( k = \ell \). For \( k \neq \ell \) we get

\[
(2^k, 0) + (0, 2^k) \in L \quad \text{whereas} \quad (2^k, 0) + (0, 2^k) \notin L,
\]

i.e., \((2^k, 0)\) and \((2^\ell, 0)\) are syntactically incongruent. Hence, \( L \) is not recognizable, and by Lemma 3.18, \( S^* \) is not a recognizable step series.
Remark 6.4. It is easy to see that recognizable trace series over commutative semirings are closed under semiring morphisms. Hence, from Example 6.3 above, we can conclude that the mc-rational trace series $([\sigma] + [\tau])^* \in N^{\langle\langle M \rangle\rangle}$ is also not recognizable. This shows that idempotency cannot be eliminated from the assumptions of Theorem 3.14 (2b), an argument already used by Droste and Gastin [7].

Example 6.5. Let $(A, \cdot)$ be the syntactic monoid of the recognizable word language $L = (\alpha \beta)^*$. We denote its five elements as follows:

\[
0 = [\alpha\alpha]_L, \quad a = [\alpha]_L, \quad b = [\beta]_L, \quad c = [\beta\alpha]_L, \quad d = [\alpha\beta]_L, \quad \text{and} \quad 1 = [\varepsilon]_L.
\]

Notice that 0 acts as an absorbing element. If we equip $A$ with the linear order $0 < a < b < c < d < 1$, then $(A, \max, \cdot, 0, 1)$ is an idempotent, finite strong bimonoid. Clearly, the trace series

\[
S = a \cdot [\sigma] + b \cdot [\tau] \in A^{\langle\langle M \rangle\rangle}
\]

is a proper and connected polynomial, and hence $S^*$ is $c$-rational. All possible factorizations of $d \in A$ using only factors from $\text{im}(S) = \{ a, b \}$ are of the form $d = (ab)^n$ with $n \geq 1$. Hence, for a trace $t \in M$ with $(S^*, t) = d$ there must be a decomposition of the form $t = ([\sigma][\tau])^n$ with $n \geq 1$. Conversely, if such a decomposition exists, then $(S^*, t) = d$. Thus, we obtain

\[
(S^*)^{-1}(d) = ([\sigma][\tau])^+.
\]

However, this trace language is not recognizable, since its preimage under $\varphi$ is the set of all non-empty words from $\Sigma^*$ containing as many $\sigma$’s as $\tau$’s. Hence, by Lemma 3.18, $S^*$ is not a recognizable step series.

7. Conclusions

Up to now, several theorems that were proved for words, like those of Kleene and Schützenberger, could be extended to traces, as Ochmański’s and Droste and Gastin’s results showed. For those reasons, traces are regarded as a very robust formalism for modeling concurrency. Since we succeeded in generalizing Droste and Vogler’s theorem from words to traces, our investigations give new evidence for this robustness. However, connectedness once more turned out to be an issue concerning the iteration of trace languages respectively trace series. Moreover, we affirmed that without idempotency of the weight structure the notion of connectedness is still too weak and mono-alphabetic connectedness is required.

Pointing in another direction, the automaton model of Droste and Gastin was very useful for proving that weighted logics and weighted asynchronous cellular automata over commutative semirings are equally expressive, cf. [12, 17, 19]. Probably, the automaton model introduced in this paper can play the same important role for the corresponding relationship in the case of weights in a strong bimonoid.

References


