The Equational Theory of Aperiodic Semigroups Is Decidable in Exponential Time

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Abstract

Due to a result by McCammond (Int. J. Algebra Comput., 2001) it is decidable whether all aperiodic semigroups satisfy some given identity of omega-terms. The respective decision procedure works by computing normal forms, but unfortunately neither its worst-case running time nor the maximal size of the intermediate terms have been estimated. We pursue a different approach and solve the same problem by means of first-order definability of regular languages and an infinite Ehrenfeucht-Fraïssé game on omega-terms. In this way, we obtain an algorithm which decides whether a given identity of omega-terms holds in all aperiodic semigroups and whose running time is exponential in the size of the omega-terms. As a byproduct we develop a framework which allows for seperating the bookkeeping involved in winning strategies for Ehrenfeucht-Fraïssé games on finite words from the actual strategy.
We fix an alphabet $\Sigma$.  

Algebra is orange.  

Logic is purple.
Semigroups and Idempotency

Definition
A semigroup is an algebraic structure $S$ with a single associative binary operation (written as juxtaposition).
An element $u \in S$ is idempotent if $uu = u$.

Examples

- Every group $G$.
  Only the neutral element is idempotent.
- $\mathcal{P}(X) = 2^X$ with union or intersection, where $X$ is an arbitrary set.
  Everything is idempotent.
- $\Sigma^+ = \text{non-empty words over } \Sigma$ with concatenation (free semigr.).
  There are no idempotents.
- $\{1, \ldots, n\}$ for $n \geq 1$ and with the operation $\oplus_n$ defined by
  \[ a \oplus_n b = \min(a + b, n). \]
  Only $n$ is idempotent.
Semigroups and Idempotency

Definition
A semigroup is an algebraic structure $S$ with a single associative binary operation (written as juxtaposition).
An element $u \in S$ is idempotent if $uu = u$.

Fact
Let $S$ be a finite semigroup.

▶ Every $u \in S$ generates a unique idempotent element $u^\omega \in S$:

$\{u^i, \ldots, u^{i+j-1}\}$ forms a subgroup of $S$ with neutral element $u^\omega$. 
Aperiodic Semigroups

Definition
A finite semigroup $S$ is aperiodic if $u^\omega u = u^\omega$ for all $u \in S$.

Examples
- The trivial group.
- $\mathcal{P}(X) = 2^X$ with union or intersection, where $X$ is an arbitrary set.
- $\{1, \ldots, n\}$ for $n \geq 1$ and with the operation $\oplus_n$ defined by
  \[
  a \oplus_n b = \min(a + b, n).
  \]

Non-Examples
- Every non-trivial finite group.
- Every finite semigroup containing a non-trivial subgroup.
Aperiodic Semigroups

Definition
A finite semigroup $S$ is aperiodic if $u^\omega u = u^\omega$ for all $u \in S$.

Fact
Let $S$ be a finite semigroup. TFAE:

1. $S$ is aperiodic.
2. For every $u \in S$ there is an $i \geq 1$ such that $u^{i+1} = u^i$.
3. $S$ contains only trivial subgroups.
Equations of \( \omega \)-Terms

Definition
The following are \( \omega \)-terms:

1. \( a \) for every \( a \in \Sigma \),
2. \( t_1 t_2 \ldots t_n \) where \( n \geq 1 \) and \( t_1, t_2, \ldots, t_n \) are \( \omega \)-terms,
3. \( (t)^\omega \) where \( t \) is an \( \omega \)-term,
4. nothing else.

Definition
Let \( S \) be a finite semigroup. The \( S \)-semantics of an \( \omega \)-term \( t \) is the induced map \( t^S : S^\Sigma \rightarrow S \) where \( t^S(\overline{u}) \) is obtained from \( t \) as follows:

1. \( a \in \Sigma \) is interpreted by \( u_a \),
2. juxtaposition — the semigroup operation,
3. \( \omega \)-power — generating the idempotent.
Equations of $\omega$-Terms

Definition
Let $S$ be a finite semigroup. The $S$-semantics of an $\omega$-term $t$ is the induced map $t^S : S^\Sigma \to S$ where $t^S(\bar{u})$ is obtained from $t$ as follows:

1. $a \in \Sigma$ is interpreted by $u_a$,
2. juxtaposition — the semigroup operation,
3. $\omega$-power — generating the idempotent.

Definition
A finite semigroup $S$ satisfies an equation $t_1 = t_2$ of $\omega$-terms $t_1$ and $t_2$ if $t_1^S = t_2^S$.

Examples
Let $S$ be a finite semigroup.

- $S$ satisfies $a^\omega a^\omega = a^\omega$ and $(a^\omega)^\omega = a^\omega$.
- $S$ satisfies $a^\omega a = a^\omega$ if and only if $S$ is aperiodic.
- $S$ satisfies $ab = ba$ if and only if $S$ is commutative.
The Main Result

Theorem (McCammond 2001 + HK 2013)

The following problem is decidable in exponential time:

Input: Two $\omega$-terms $t_1$ and $t_2$.

Question: Does every finite aperiodic semigroup satisfy $t_1 = t_2$?

Proof Sketch.

1. Use the correspondence between aperiodic finite semigroups and the class of first-order definable languages.
2. Investigate the infinite Ehrenfeucht-Fraïssé game on labelled linear orderings.
3. Assign to every $\omega$-term $t$ a labelled linear ordering $\llbracket t \rrbracket_\varrho$ and introduce an Ehrenfeucht-Fraïssé game on $\omega$-terms.
4. Show that $t_1 = t_2$ is satisfied by every aperiodic finite semigroup if and only if $\llbracket t_1 \rrbracket_\varrho = \llbracket t_2 \rrbracket_\varrho$.
5. The latter is decidable in exponential time.
The Syntactic Semigroup

Definition
Let $L \subseteq \Sigma^+$ be a language over $\Sigma$. The syntactic congruence of $L$ is the binary relation $\equiv_L$ on $\Sigma^+$ defined by

$$u \equiv_L v \iff \forall x, y \in \Sigma^*: (xuy \in L \iff xvy \in L).$$

The syntactic semigroup of $L$ is the quotient semigroup $S_L := \Sigma^+ / \equiv_L$.

Theorem (Myhill 1958, Rabin/Scott 1959)
Let $L \subseteq \Sigma^+$ be a language. TFAE:

1. $L$ is regular.
2. The syntactic semigroup $S_L$ of $L$ is finite.
First-Order Logic over Words

Definition
The syntax of first-order logic (FO) over words is given by

\[ \varphi ::= x = y \mid x \leq y \mid \lambda(x) = a \mid \varphi \lor \varphi \mid \varphi \land \varphi \mid \neg \varphi \mid \exists x \varphi \mid \forall x \varphi \]

where \( x \) and \( y \) are from a fixed infinite set of variables and \( a \in \Sigma \).

Definition
A word \( w \in \Sigma^+ \) is a model of a sentence \( \varphi \in \text{FO} \), in symbols \( w \models \varphi \), if \( w \) satisfies \( \varphi \) under the following assumptions:

1. \( w \) is regarded as a (finite) \( \Sigma \)-labelled linear ordering, e.g.,
   \[ \text{ailogic} = a \rightarrow i \rightarrow l \rightarrow o \rightarrow g \rightarrow i \rightarrow c, \]

2. variables \( x, y, \ldots \) range over the positions of \( w \),
3. \( \leq \) is interpreted w.r.t. the linear ordering of positions,
4. \( \lambda \) is interpreted by the labelling map.
First-Order Definable Languages

Definition
Let $\varphi \in \text{FO}$ be a sentence. The language defined by $\varphi$ is

$$L(\varphi) = \{ w \in \Sigma^+ \mid w \models \varphi \}.$$

Corollary (Büchi/Elgot 1958, Trakhtenbrot 1961)
The language $L(\varphi)$ is regular for every sentence $\varphi \in \text{FO}$.

Theorem (Schützenberger 1965 + McNaughton/Papert 1971)
Let $L \subseteq \Sigma^+$ be a language. TFAE:

1. The syntactic semigroup $S_L$ of $L$ is finite and aperiodic.
2. $L$ is first-order definable, i.e., $L = L(\varphi)$ for some sentence $\varphi \in \text{FO}$. 
The Main Result

Theorem (McCammond 2001 + HK 2013)

The following problem is decidable in exponential time:

Input: Two ω-terms $t_1$ and $t_2$.

Question: Does every finite aperiodic semigroup satisfy $t_1 = t_2$?

Proof Sketch.

1. Use the correspondence between aperiodic finite semigroups and the class of first-order definable languages.

2. Investigate the infinite Ehrenfeucht-Fraïssé game on labelled linear orderings.

3. Assign to every ω-term $t$ a labelled linear ordering $\llbracket t \rrbracket_\omega$ and introduce an Ehrenfeucht-Fraïssé game on ω-terms.

4. Show that $t_1 = t_2$ is satisfied by every aperiodic finite semigroup if and only if $\llbracket t_1 \rrbracket_\omega = \llbracket t_2 \rrbracket_\omega$.

5. The latter is decidable in exponential time.
Generalised Words

Definition
A generalised word is a non-empty, countable $\Sigma$-labelled linear ordering. The set of generalised words with concatenation forms a semigroup.

First-order logic (FO) is straightforwardly extended to generalised words.

Examples

ailogic = $a \rightarrow i \rightarrow l \rightarrow o \rightarrow g \rightarrow i \rightarrow c$,

$(ai)^N = aiaiai \ldots = a \rightarrow i \rightarrow a \rightarrow i \rightarrow a \rightarrow i \ldots \ldots \ldots$,

$a^{-N} = \ldots aaaaa = \ldots \ldots \ldots \rightarrow a \rightarrow a \rightarrow a \rightarrow a \rightarrow a$,

$b^Z = \ldots bbb \ldots = \ldots \ldots \rightarrow a \rightarrow a \rightarrow a \rightarrow a \rightarrow a \ldots \ldots \ldots \rightarrow = b^{-N}b^N$,

$(ab)^Q = \ldots \rightarrow a \rightarrow b \rightarrow \ldots \rightarrow a \rightarrow b \rightarrow \ldots \rightarrow a \rightarrow b \rightarrow \ldots \ldots \ldots \rightarrow$.
The Ehrenfeucht-Fraïssé Game on Generalised Words

Definition
The Ehrenfeucht-Fraïssé game $\text{EF}_n(u, v)$ is as follows:

Players: Spoiler and Duplicator.

Board: Two generalised words $u$ and $v$.

Rounds: $n \geq 0$.

$i^{\text{th}}$ Round: Spoiler chooses a position $p_i$ in $u$ or $q_i$ in $v$; Duplicator chooses a position $q_i$ in $v$ or $p_i$ in $u$.

Winner: Duplicator if the sequences $p_1 p_2 \ldots p_n$ (in $u$) and $q_1 q_2 \ldots q_n$ (in $v$) are ordered and labelled the same way; Spoiler otherwise.

Theorem (Fraïssé 1954, Ehrenfeucht 1961)
Let $u$ and $v$ be generalised words and $n \geq 0$. TFAE:

1. Duplicator has a winning strategy in $\text{EF}_n(u, v)$.
2. $u \models \varphi$ precisely if $v \models \varphi$, for all sentences $\varphi \in \text{FO}$ with $\text{qd}(\varphi) \leq n$. 
The Infinite Ehrenfeucht-Fraïssé Game on Generalised Words

Definition
The infinite Ehrenfeucht-Fraïssé game $\text{EF}_\infty(u, v)$ is as follows:

- **Players:** Spoiler and Duplicator.
- **Board:** Two generalised words $u$ and $v$.
- **Rounds:** Infinitely many.

$i$th Round: Spoiler chooses a position $p_i$ in $u$ or $q_i$ in $v$; Duplicator chooses a position $q_i$ in $v$ or $p_i$ in $u$.

**Winner:** Duplicator if the sequences $p_1p_2p_3\ldots$ (in $u$) and $q_1q_2q_3\ldots$ (in $v$) are ordered and labelled the same way; Spoiler otherwise.

Question
Let $u$ and $v$ be generalised words. How are the following related?

1. **Duplicator** has a winning strategy in $\text{EF}_n(u, v)$ for every $n \geq 0$.
2. **Duplicator** has a winning strategy in $\text{EF}_\infty(u, v)$. 
Definition
The infinite Ehrenfeucht-Fraïssé game $\text{EF}_\infty(u, v)$ is as follows:

 Players: Spoiler and Duplicator.
 Board: Two generalised words $u$ and $v$.
 Rounds: Infinitely many.
 $i^{th}$ Round: Spoiler chooses a position $p_i$ in $u$ or $q_i$ in $v$; Duplicator chooses a position $q_i$ in $v$ or $p_i$ in $u$.
 Winner: Duplicator if the sequences $p_1p_2p_3\ldots$ (in $u$) and $q_1q_2q_3\ldots$ (in $v$) are ordered and labelled the same way; Spoiler otherwise.

Example
Consider the generalised words $u = a^Z$ and $v = a^Za^Z$. Then:

1. Duplicator has a winning strategy in $\text{EF}_n(u, v)$ for every $n \geq 0$.
2. Spoiler has a winning strategy in $\text{EF}_\infty(u, v)$.
**$\varrho$-Rational Generalised Words**

**Definition**
The linear ordering $\varrho = \mathbb{N} + \mathbb{Q} \times \mathbb{Z} + (\mathbb{N})$ is

![Diagram](image)

**Definition**
Let $u$ be a generalised word. The $\varrho$-power of $u$ is the generalised word $u^\varrho$ which is obtained from $\varrho$ by replacing every position in $\varrho$ with $u$.

**Definition**
A generalised word is $\varrho$-rational if it can be constructed from the finite words using concatenation and $\varrho$-power.

**Example**
The words $a^Z$ and $a^Z a^Z$ are **not** $\varrho$-rational.
The Limit Strategy

Lemma
Let \( u \) and \( v \) be \( \varrho \)-rational generalised words. TFAE:

1. Duplicator has a winning strategy in \( \text{EF}_n(u, v) \) for every \( n \geq 0 \).
2. Duplicator has a winning strategy in \( \text{EF}_\infty(u, v) \).

Proof of \( 1 \Rightarrow 2 \).

\[ \begin{align*}
\text{Proof of } 1 \Rightarrow 2. \\
\text{Assume the game is in the } i^{\text{th}} \text{ round and Duplicator still has a winning strategies for the next } n \text{ rounds for every } n \geq 0. \\
\text{W.l.o.g., Spoiler chooses a position } p_i \text{ in } u. \\
\text{For each } n \geq 1, \text{ let position } q_i^{(n)} \text{ in } v \text{ be Duplicator’s answer in her winning strategy for the next } n \text{ rounds.} \\
\text{Based on the } \varrho \text{-rationality of } v, \text{ one can define a “limit position” } q_i = \lim_{n \to \infty} q_i^{(n)} \\
\text{such that after choosing position } q_i \text{ in } v, \text{ Duplicator still has winning strategies for the next } n \text{ rounds for every } n \geq 0. \quad \Box
\end{align*} \]
The Infinite Ehrenfeucht-Fraïssé Game on Generalised Words

Definition
The infinite Ehrenfeucht-Fraïssé game $\text{EF}_\infty(u, v)$ is as follows:

- **Players:** Spoiler and Duplicator.
- **Board:** Two generalised words $u$ and $v$.
- **Rounds:** Infinitely many.

$i^{th}$ Round:
- **Spoiler** chooses a position $p_i$ in $u$ or $q_i$ in $v$;
- **Duplicator** chooses a position $q_i$ in $v$ or $p_i$ in $u$.

**Winner:** Duplicator if the sequences $p_1p_2p_3\ldots$ (in $u$) and $q_1q_2q_3\ldots$ (in $v$) are ordered and labelled the same way; Spoiler otherwise.

Theorem (HK 2013)
Let $u$ and $v$ be $\varrho$-rational generalised words. TFAE:

1. Duplicator has a winning strategy in $\text{EF}_\infty(u, v)$.
2. $u \models \varphi$ precisely if $v \models \varphi$, for all sentences $\varphi \in \text{FO}$.
The Main Result

Theorem (McCammond 2001 + HK 2013)

The following problem is decidable in exponential time:

**Input:** Two $\omega$-terms $t_1$ and $t_2$.

**Question:** Does every finite aperiodic semigroup satisfy $t_1 = t_2$?

Proof Sketch.

1. Use the correspondence between aperiodic finite semigroups and the class of first-order definable languages.
2. Investigate the infinite Ehrenfeucht-Fraïssé game on labelled linear orderings.
3. Assign to every $\omega$-term $t$ a labelled linear ordering $[t]_\varphi$ and introduce an Ehrenfeucht-Fraïssé game on $\omega$-terms.
4. Show that $t_1 = t_2$ is satisfied by every aperiodic finite semigroup if and only if $[t_1]_\varphi = [t_2]_\varphi$.
5. The latter is decidable in exponential time.
The Infinite Ehrenfeucht-Fraïssé Game on $\omega$-Terms

Definition
The $\varrho$-semantics of an $\omega$-term $t$ is the generalised word $[t]_\varrho$ obtained from $t$ as follows:

1. $a \in \Sigma$ is interpreted by the one letter word $a$,
2. juxtaposition —— concatenation,
3. $\omega$-power —— $\varrho$-power.

Observation
A generalised word is $\varrho$-rational if and only if it is the $\varrho$-semantics of some $\omega$-term.

Theorem (HK 2013)
Let $t_1$ and $t_2$ be $\omega$-terms. TFAE:

1. Duplicator has a winning strategy in $\text{EF}_\infty([t_1]_\varrho, [t_2]_\varrho)$.
2. The syntactic semigroup $S_L$ of every first-order definable language $L \subseteq \Sigma^+$ satisfies $t_1 = t_2$. 
Proof of the Ehrenfeucht-Fraïssé Theorem for $\omega$-Terms

Theorem (HK 2013)

Let $t_1$ and $t_2$ be $\omega$-terms. TFAE:

1. Duplicator has a winning strategy in $\text{EF}_\infty([t_1]_\varrho, [t_2]_\varrho)$.
2. The syntactic semigroup $S_L$ of every first-order definable language $L \subseteq \Sigma^+$ satisfies $t_1 = t_2$.

Definition

Let $k \geq 1$. The $k$-semantics of an $\omega$-term $t$ is the finite word $[[t]]_k \in \Sigma^+$ which is obtained from $t$ almost like the $\varrho$-semantics except that

3. $\omega$-power is interpreted by $k^{\text{th}}$ power.

Lemma

Let $t_1$ and $t_2$ be $\omega$-terms, $n \geq 0$ and $k \geq 2^{n+1} - 1$. TFAE:

1. Duplicator has a winning strategy in $\text{EF}_n([t_1]_k, [t_2]_k)$.
2. Duplicator has a winning strategy in $\text{EF}_n([t_1]_\varrho, [t_2]_\varrho)$.
Application of the Ehrenfeucht-Fraïssé Theorem for $\omega$-Terms

Theorem (HK 2013)

Let $t_1$ and $t_2$ be $\omega$-terms. TFAE:

1. Duplicator has a winning strategy in $\text{EF}_\infty([t_1]_\varrho, [t_2]_\varrho)$.
2. The syntactic semigroup $S_L$ of every first-order definable language $L \subseteq \Sigma^+$ satisfies $t_1 = t_2$.

Corollary (Schützenberger 1965 + McNaughton/Papert 1971)

The syntactic semigroup $S_L$ of every first-order definable language $L \subseteq \Sigma^+$ is aperiodic.

Proof.

- $S_L$ is finite by the Büchi-Elgot-Trakhtenbrot theorem.
- We have $[a^\omega a]_\varrho = a^\varrho a = a^\varrho = [a^\omega]_\varrho$.
- Duplicator has a winning strategy in $\text{EF}_\infty([a^\omega a]_\varrho, [a^\omega]_\varrho)$.
- $S_L$ satisfies $a^\omega a = a^\omega$, i.e., $S_L$ is aperiodic.  \qed
The Main Result

Theorem (McCammond 2001 + HK 2013)

The following problem is decidable in exponential time:

Input: Two $\omega$-terms $t_1$ and $t_2$.

Question: Does every finite aperiodic semigroup satisfy $t_1 = t_2$?

Proof Sketch.

1. Use the correspondence between aperiodic finite semigroups and the class of first-order definable languages.
2. Investigate the infinite Ehrenfeucht-Fraïssé game on labelled linear orderings.
3. Assign to every $\omega$-term $t$ a labelled linear ordering $[t]_\varrho$ and introduce an Ehrenfeucht-Fraïssé game on $\omega$-terms.
4. Show that $t_1 = t_2$ is satisfied by every aperiodic finite semigroup if and only if $[t_1]_\varrho = [t_2]_\varrho$.
5. The latter is decidable in exponential time.
The Reduction

Proposition

Let $t_1$ and $t_2$ be $\omega$-terms. TFAE:

1. $[t_1]_\varrho = [t_2]_\varrho$.
2. Every syntactic aperiodic finite semigroup satisfies $t_1 = t_2$.
3. Every aperiodic finite semigroup satisfies $t_1 = t_2$.
4. $t_1 = t_2$ can be deduced from the following axioms, where $n \geq 1$:
   
   \[
   \begin{align*}
   (ab)c &= a(bc) & (a^\omega)^\omega &= a^\omega & (a^n)^\omega &= a^\omega \\
   a^\omega a^\omega &= a^\omega & a^\omega a &= aa^\omega = a^\omega & (ab)^\omega a &= a(ba)^\omega.
   \end{align*}
   \]

Proof.

1 $\Rightarrow$ 2: Follows from the Ehrenfeucht-Fraïssé theorem for $\omega$-terms.

2 $\Rightarrow$ 3: Follows directly from Eilenberg’s variety theorem.

3 $\Rightarrow$ 4: Part of McCammond’s results.

4 $\Rightarrow$ 1: The semigroup of generalised words with $\omega$-power interpreted by $\varrho$-power also satisfies the axioms (up to isomorphism). $\square$
The Main Result

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3. Assign to every $\omega$-term $t$ a labelled linear ordering $\langle t \rangle_\$ and introduce an Ehrenfeucht-Fraïssé game on $\omega$-terms.
4. Show that $t_1 = t_2$ is satisfied by every aperiodic finite semigroup if and only if $\langle t_1 \rangle_\$ = $\langle t_2 \rangle_\$.
5. The latter is decidable in exponential time.
The Isomorphism Problem for Regular Generalised Words

Definition
A generalised word is regular if it can be constructed from the finite words using concatenation, \( \mathbb{N} \)-power, \((-\mathbb{N})\)-power, \( \mathbb{Z} \)-power, \( \mathbb{Q} \)-power and dense shuffle.

Observations
Since \( u^{\varrho} = u^{\mathbb{N}}(u^{\mathbb{Z}})^{\mathbb{Q}}u^{-\mathbb{N}} \) holds for every generalised word \( u \), we obtain:

1. All \( \varrho \)-rational generalised words are regular.
2. The translation from an \( \omega \)-term \( t \) to a regular expressions for \( \llbracket t \rrbracket_{\varrho} \) involves an exponential blow-up.

Theorem (Bloom/Esik 2005)

The following problem is decidable in polynomial time:

Input: Two generalised words \( u \) and \( v \) as regular expressions.

Question: \( u = v \)?
The Isomorphism Problem for Regular Generalised Words

Observations
Since \( u^\varrho = u^N (u^Z)^Q u^{-N} \) holds for every generalised word \( u \), we obtain:

1. All \( \varrho \)-rational generalised words are regular.
2. The translation from an \( \omega \)-term \( t \) to a regular expressions for \( \llbracket t \rrbracket_\varrho \) involves an exponential blow-up.

Theorem (Bloom/Esik 2005)

The following problem is decidable in polynomial time:

- Input: Two generalised words \( u \) and \( v \) as regular expressions.
- Question: \( u = v \)?

Corollary

The following problem is decidable in exponential time:

- Input: Two \( \omega \)-terms \( t_1 \) and \( t_2 \).
- Question: \( \llbracket t_1 \rrbracket_\varrho = \llbracket t_2 \rrbracket_\varrho \)?
The Main Result

Theorem (McCammond 2001 + HK 2013)

The following problem is decidable in exponential time:

Input: Two \( \omega \)-terms \( t_1 \) and \( t_2 \).

Question: Does every finite aperiodic semigroup satisfy \( t_1 = t_2 \)?

Proof Sketch.

1. Use the correspondence between aperiodic finite semigroups and the class of first-order definable languages.

2. Investigate the infinite Ehrenfeucht-Fraïssé game on labelled linear orderings.

3. Assign to every \( \omega \)-term \( t \) a labelled linear ordering \( \llbracket t \rrbracket_\varphi \) and introduce an Ehrenfeucht-Fraïssé game on \( \omega \)-terms.

4. Show that \( t_1 = t_2 \) is satisfied by every aperiodic finite semigroup if and only if \( \llbracket t_1 \rrbracket_\varphi = \llbracket t_2 \rrbracket_\varphi \).

5. The latter is decidable in exponential time.
Extensions and Open Problems

Theorem (HK 2013)

Let $\mathcal{F} \subseteq \text{FO}$ be a fragment of first-order logic satisfying certain natural syntactic closure properties.

Let $t_1$ and $t_2$ be $\omega$-terms. TFAE:

1. Duplicator has a winning strategy in $\text{EF}_\infty(\langle t_1 \rangle_\varphi, \langle t_2 \rangle_\varphi)$.
2. The syntactic semigroup $S_L$ of every $\mathcal{F}$-definable language $L \subseteq \Sigma^+$ satisfies $t_1 = t_2$.

Open Problem

Are the two (equivalent) problems below decidable in polynomial time?

Input: Two $\omega$-terms $t_1$ and $t_2$.

Question 1: Does Duplicator have a winning strategy in $\text{EF}_\infty(\langle t_1 \rangle_\varphi, \langle t_2 \rangle_\varphi)$?

Question 2: Does every aperiodic finite semigroup satisfy $t_1 = t_2$?