On Threshold Logic and Cutting Planes Proofs

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Yet Another Formulation of Propositional Threshold Logic

Let PTK^* be defined like PTK in [1, 2], but with the rule T_k^n-right replaced by the two rules

$$T_k^n\text{-right1}: \frac{\Gamma \Longrightarrow A_1, \Delta \qquad \Gamma \Longrightarrow T_{k-1}^{n-1}(A_2, \ldots, A_n), \Delta}{\Gamma \Longrightarrow T_k^n(A_1, \ldots, A_n), \Delta}$$

$$T_k^n\text{-right2}: \frac{\Gamma \Longrightarrow T_k^{n-1}(A_2, \ldots, A_n), \Delta}{\Gamma \Longrightarrow T_k^n(A_1, \ldots, A_n), \Delta}$$

and T_k^n-left replaced by the two dual rules

$$T_k^n\text{-left1}: \frac{A_1, \Gamma \Longrightarrow \Delta \qquad T_k^{n-1}(A_2, \ldots, A_n), \Gamma \Longrightarrow \Delta}{T_k^n(A_1, \ldots, A_n), \Gamma \Longrightarrow \Delta}$$

$$T_k^n\text{-left2}: \frac{T_{k-1}^{n-1}(A_2, \ldots, A_n), \Gamma \Longrightarrow \Delta}{T_k^n(A_1, \ldots, A_n), \Gamma \Longrightarrow \Delta} .$$

The correctness of PTK^* is obvious, and the completeness follows from Theorem 1 below and the completeness of PTK. In the following, we show that PTK and PTK^* are polynomially equivalent, and that the mutual simulations also respect the depth of proofs. This was claimed without proof in [3], where PTK^* was first defined.

Theorem 1. *If P is a proof in PTK, then there is a proof P' in PTK^* of the same end-sequent. The size of P' is linear in the size of P, and the formula depths of P and P' are the same.*

Proof. Each application of the rule T_k^n-right is replaced by a subproof that is built as follows: From the second premise we get by weakening the sequent

$$\Gamma \Longrightarrow T_{k-1}^{n-1}(A_2, \ldots, A_n), T_k^{n-1}(A_2, \ldots, A_n), \Delta \ ,$$

and from this and the first premise we get by an application of T_k^n-right1

$$\Gamma \implies T_k^n(A_1, \dots, A_n), T_k^{n-1}(A_2, \dots, A_n), \Delta \ .$$

From this sequent we obtain the conclusion by structural inferences and T_k^n-right2. Likewise, each application of T_k^n-left is replaced by a similar, dual subproof. The size and depth bounds are obvious. □

Theorem 2. *If P is a proof in PTK^*, then there is a proof P' in PTK of the same end-sequent. The size of P' is polynomial in the size of P, and the formula depths of P and P' are the same.*

Proof. First, each application of the rule T_k^n-right1 can be simulated by T_k^n-right of PTK preceded by a weakening, and likewise T_k^n-left1 can be simulated using weakening and T_k^n-left.

In [2] it was noted that the sequents

$$(*) \qquad T_\ell^m(A_1, \dots, A_m) \implies T_{\ell-1}^m(A_1, \dots, A_m)$$

have proofs in PTK of size polynomial in m. Using these, we can replace each application of T_k^n-right2 by a subproof constructed as follows: From the premise of T_k^n-right2 and an instance of $(*)$ we obtain

$$\Gamma \implies T_{k-1}^{n-1}(A_2, \dots, A_n), \Delta \ ,$$

by a cut, and again from the premise of T_k^n-right2 we obtain by weakening

$$\Gamma \implies A_1, T_k^{n-1}(A_2, \dots, A_n), \Delta \ .$$

From these two we obtain the conclusion by T_k^n-right. A dual proof using $(*)$ can serve to replace applications of T_k^n-left2. The size bound holds if we see the two uses of the premise of T_k^n-right2 as identical, i.e. if the proof is not tree-like. □

Theorems 1 and 2 together imply that PTK^* enjoys cut-elimination, as the subproofs used in the proof of Theorem 1 are cut-free. They are also tree-like, hence Theorem 1 also holds for cut-free and tree-like proofs. The subproofs used in the proof of Theorem 2 are, as noted, not tree-like, and use cuts. Hence a question is:

> Do cut-free and/or tree-like PTK-proofs polynomially simulate cut-free / tree-like PTK^*-proofs?

Another problem is to improve the size bounds in Theorem 2.

Embedding Unary Cutting Planes into PTK^*

A Unary Cutting Planes (CP^*) inequality can be written in the form

$$\sum_{i=1}^{n} x_i - \sum_{i=n+1}^{n+m} x_i \geq k \ ,$$

where $n, m \in \mathbb{N}$, $k \in \mathbb{Z}$ and the variables $x_1, \ldots, x_{n+m}$ are not necessarily distinct. By a result in [2], a CP^*-proof can be assumed to use only the axioms $x \geq 0$, $-x \geq -1$, addition and division by 2.

For convenience, let $T_0^n(A_1, \ldots, A_n)$ for $n \geq 0$ stand for $\top$, and $T_k^0()$ with $k > 0$ stand for $\bot$. Let E denote the inequality above, then its translation $\hat{E}$ in PTK is defined as

$$T_r^{n+m}(x_1, \ldots, x_n, \neg x_{n+1}, \ldots, \neg x_{n+m}) \ ,$$

where $r := \max(k + m, 0)$.

Theorem 3. *Let P be a CP^*-proof of an inequality E from the inequalities $E_1, \ldots, E_n$. Then there is a PTK^*-proof of the sequent*

$$\hat{E}_1, \ldots, \hat{E}_n \implies \hat{E}$$

of threshold depth 1 and size $O(|P|^{O(1)})$.

This implies that threshold depth 1 PTK^*-proofs can p-simulate CP^* in the following sense:

Corollary 4. *If A is a tautology in DNF such that $\neg A$, written as a set of CP^*-inequalities, has a CP^*-refutation of size s, then there is a PTK^*-proof of A of threshold depth 1 and size $O(s^{O(1)} + |A|)$.*

Proof. Let A be $\bigvee_{i \leq n} \bigwedge_{j \in J_i} \ell_{ij}$, then by the theorem there is a proof in PTK^* of

$$\bigvee_{j \in J_1} \bar{\ell}_{1j}, \ldots, \bigvee_{j \in J_n} \bar{\ell}_{nj} \implies \bot$$

of threshold depth 1 and size $O(s^{O(1)})$. From this, A can be derived trivially in size $O(|A|)$. □

By Theorem 2, the same holds for PTK in place of PTK^*. To prove Theorem 3, we first derive a series of lemmas. The first lemma is simple and can be proved by the reader.

Lemma 5. *There is a proof in PTK* of the sequent*

$$T^n_k(A_1,\dots,A_n) \implies T^n_{k-1}(A_1,\dots,A_n)$$

of threshold depth 1 and size $O(n)$

Here, as well as in the following lemmas, when we say a proof has threshold depth 1 we mean that its threshold depth is at most 1 + the maximal threshold depth of the subformulae A_i. In particular, its threshold depth is 1 if the A_i do not contain any threshold connectives.

Lemma 6. *There is a proof in PTK* of the equivalence*

$$T^{n+2}_{k+1}(A,\neg A,B_1,\dots,B_n) \leftrightarrow T^n_k(B_1,\dots,B_n)$$

of threshold depth 1 and size $O(n)$.

Proof. Let $\vec{B}$ abbreviate $B_1,\dots,B_n$. From the axioms $T^n_k(\vec{B}) \implies T^n_k(\vec{B})$ and $A \implies A$, we get the sequent

$$T^{n+2}_{k+1}(A,\neg A,\vec{B}) \implies A, T^n_k(\vec{B})$$

by T^n_k-left2 and then T^n_k-left1. In the same way using the axiom $\neg A \implies \neg A$ we get

$$T^{n+2}_{k+1}(A,\neg A,\vec{B}) \implies \neg A, T^n_k(\vec{B})$$

using T^n_k-left1 first and then T^n_k-left2. From these the sequent in the lemma follows by a cut. □

Lemma 7. *There is a proof in PTK* of the following equivalence, the* generalized De Morgan law

$$\neg T^n_k(A_1,\dots,A_n) \leftrightarrow T^n_{n-k+1}(\neg A_1,\dots,\neg A_n)$$

of threshold depth 1 and size $O(n^3)$.

Proof. For the direction from left to right, we have to derive the sequent $S_{n,k} := \implies T^n_k(A_1,\dots,A_n), T^n_{n-k+1}(\neg A_1,\dots,\neg A_n)$. First, we derive $S_{n,n}$: From the sequents $\implies A_i, \neg A_i$ for $1 \le i \le n$, this is obtained by $\wedge$-right followed by $\vee$-right. Dually we get $S_{n,1}$.

Now for $1 < k < n$, we derive $S_{n,k}$ from $S_{n-1,k}$ and $S_{n-1,k-1}$ as follows: From $\implies T^{n-1}_{k-1}(A_2,\dots,A_n), T^{n-1}_{n-k+1}(\neg A_2,\dots,\neg A_n)$ and the axiom $A_1 \implies A_1$, we derive

$$A_1 \implies T^n_k(A_1,\dots,A_n), T^n_{n-k+1}(\neg A_1,\dots,\neg A_n)$$

by T_k^n-right1 and then T_k^n-right2. Likewise, from the axiom $\neg A_1 \implies \neg A_1$ and $\implies T_k^{n-1}(A_2, \ldots, A_n), T_{n-k}^{n-1}(\neg A_2, \ldots, \neg A_n)$ we derive

$$\neg A_1 \implies T_k^n(A_1, \ldots, A_n), T_{n-k+1}^n(\neg A_1, \ldots, \neg A_n) \; .$$

From these, $S_{n,k}$ is obtained by a cut.

Now a proof for $S_{n,k}$ is obtained by arranging the sequents $S_{i+j,i}$ for $1 \leq i \leq k$ and $0 \leq j \leq n-k$ in a rectangular matrix, where each sequent is proved from those to the left and above it, and those in the first row and column are derived directly. Thus, we get a proof of the direction from left to right that has $O(n^2)$ many sequents and is hence of size $O(n^3)$.

The direction from right to left is proved dually. □

Lemma 8. *For each permutation $\pi \in S_n$, there is a proof in PTK^* of the sequent*

$$T_k^n(A_1, \ldots, A_n) \implies T_k^n(A_{\pi(1)}, \ldots, A_{\pi(n)})$$

of threshold depth 1 and size $O(n^4)$.

Proof. We start by proving that the sequents

$$(*) \qquad T_k^n(A, B, \vec{C}) \implies T_k^n(B, A, \vec{C})$$

have proofs of threshold depth 1 and size $O(n)$. First, using the axioms $T_{k-2}^{n-2}(\vec{C}) \implies T_{k-2}^{n-2}(\vec{C})$ as well as $A \implies A$ and $B \implies B$ we derive

$$\tilde{A}, \tilde{B}, T_k^n(A, B, \vec{C}) \implies T_k^n(B, A, \vec{C})$$

for each choice of $\tilde{A} = A$ or $\neg A$ and $\tilde{B} = B$ or $\neg B$, which is easily done. From these, $(*)$ is obtained by several cuts. This proof uses constantly many steps, hence is of size $O(n)$.

Next we prove the lemma for special permutations consisting of one cycle of the form $(p\, p{-}1\, \ldots\, 1)$: the sequents

$$(**) \qquad T_k^n(A_1, \ldots, A_n) \implies T_k^n(A_p, A_1, \ldots, A_{p-1}, A_{p+1}, \ldots, A_n)$$

have proofs of threshold depth 1 and size $O(n^3)$. Note that the sequent $(**)$ is easily derived for $k = n$ and $k = 1$ using structural inferences, and for $p = 2$ it is just an instance of the sequent $(*)$ above.

Next we derive $(**)$ from the two sequents

$$T_j^{n-1}(A_2, \ldots, A_n) \implies T_j^{n-1}(A_p, A_2, \ldots, A_{p-1}, A_{p+1}, \ldots, A_n)$$

for $j = k, k-1$ and $A_1 \implies A_1$, using first the T^n_k-rules and a cut to add A_1 on both sides, and then an instance of $(*)$ and a cut to swap A_1 and A_p in the succedent.

Using these, an inductive proof of $(**)$ can be built as a rectangular matrix as in the proof of Lemma 7, and like there the size of the resulting proof will be $O(n^3)$.

For the general case, note that any permutation $\pi \in S_n$ can be factored into at most n cycles of the above type, hence we get a proof for a general π by at most $n-1$ cuts from instances of the special case above, which gives a proof of size $O(n^4)$. □

Lemma 9. *The rule T^n_k-right2 of PTK′*

$$\frac{\Gamma \implies T^n_k(A_1,\ldots,A_n),\Delta \qquad \Gamma \implies T^m_\ell(B_1,\ldots,B_m),\Delta}{\Gamma \implies T^{n+m}_{k+\ell}(A_1,\ldots,A_n,B_1,\ldots,B_m),\Delta}$$

can be simulated in PTK by a proof of threshold depth 1 and size $O(m^2(m+n)^4)$.*

Proof. We give a proof of the sequent $S_{m,\ell}$ defined as

$$T^n_k(A_1,\ldots,A_n), T^m_\ell(B_1,\ldots,B_m) \implies T^{n+m}_{k+\ell}(A_1,\ldots,A_n,B_1,\ldots,B_m)\ ,$$

then the claim follows by using cuts. First we derive the sequents $S_{m,m}$ from the axioms $T^n_k(A_1,\ldots,A_n) \implies T^n_k(A_1,\ldots,A_n)$ and $B_i \implies B_i$ for $1 \le i \le m$ giving

$$T^n_k(A_1,\ldots,A_n), T^m_m(B_1,\ldots,B_m) \implies T^{n+m}_{k+m}(B_m,\ldots,B_1,A_1,\ldots,A_n)$$

from which we get $S_{m,m}$ by Lemma 8. The size of this proof is dominated by the size of the proof from Lemma 8, hence it is of size $O((m+n)^4)$.

Similarly from $T^n_k(A_1,\ldots,A_n) \implies T^n_k(A_1,\ldots,A_n)$ and $B_i \implies B_i$, we get

$$T^n_k(A_1,\ldots,A_n), B_i \implies T^{n+m}_{k+1}(A_1,\ldots,A_n,B_1,\ldots,B_m)$$

for each $1 \le i \le m$, hence a $\vee$-left yields $S_{m,1}$. This proof consists of m subproofs, each using a proof obtained from Lemma 8, so it is of size $O(m(m+n)^4)$.

Now we show how to derive $S_{m,\ell}$ from $S_{m-1,\ell-1}$ and $S_{m-1,\ell}$, then a proof of $S_{m,\ell}$ is built as in the proof of Lemma 7. First from $S_{m-1,\ell}$ (with the variables $B_2,\ldots,B_m$) and $B_1 \implies B_1$ we obtain

$$T^n_k(A_1,\ldots,A_n), T^m_\ell(B_1,\ldots,B_m) \implies B_1, T^{n+m}_{k+\ell}(A_1,\ldots,A_n,B_1,\ldots,B_m)$$

On the other hand, from $S_{m-1,\ell-1}$ and $B_1 \implies B_1$ we obtain

$$T_k^n(A_1, \dots, A_n), T_\ell^m(B_1, \dots, B_m), B_1 \implies T_{k+\ell}^{n+m}(A_1, \dots, A_n, B_1, \dots, B_m)$$

Hence we obtain $S_{m,\ell}$ by a cut.

The whole proof of $S_{m,\ell}$ consists of $O(m^2)$ many proofs of size $O((m+n)^4)$, plus $O(m)$ proofs of sequents $S_{i,i}$ and $S_{i,1}$ whose size is negligible, hence its size is $O(m^2(m+n)^4)$. □

Proof of Theorem 3. By induction on the number of inferences in P. If this number is 1, then P consists only of the inequality E, and either $E = E_i$ for some $1 \leq i \leq n$, or E is a CP^*-axiom $x \geq 0$ or $-x \geq -1$. In either of these cases, the claim is trivial. Otherwise, P has a last inference, and we have to distinguish whether this is an addition or a division inference.

Let the last inference be an addition whose premises are

$$\sum_{i=1}^{n} x_i - \sum_{i=n+1}^{n+m} x_i \geq k \quad \text{and} \quad \sum_{i=1}^{p} y_i - \sum_{i=p+1}^{p+q} y_i \geq \ell$$

and whose conclusion is

$$\sum_{i=1}^{s} z_i - \sum_{i=n+1}^{s+t} z_i \geq k + \ell \ ,$$

with $s = n + p - c$ and $t = m + q - c$, where c is the number of cancellations in the inference. We treat only the case where $k + m \geq 0$ and $\ell + q \geq 0$. So from the translations of the premises we get by Lemma 9

$$T_{k+\ell+m+q}^{n+m+p+q}(x_1, \dots, x_n, \neg x_{n+1}, \dots, \neg x_{n+m}, y_1, \dots, y_p, \neg y_{p+1}, \dots, \neg y_{p+q}) \ .$$

By Lemma 8 we can sort the arguments such that all possible cancellations can be made by c applications of Lemma 6. After that the arguments can be sorted using Lemma 8 such that the result is

$$T_{k+\ell+t}^{s+t}(z_1, \dots, z_s, \neg z_{s+1}, \dots, \neg z_{s+t}) \ ,$$

which is the translation of the conclusion of the addition inference.

For the case of division, suppose we have

$$T_k^{2n}(A_1, A_1, A_2, A_2, \dots, A_n, A_n) \ .$$

We want to derive $T^n_{\lceil\frac{k}{2}\rceil}(A_1, A_2, \ldots, A_n)$, so for sake of contradiction, assume $\neg T^n_{\lceil\frac{k}{2}\rceil}(A_1, A_2, \ldots, A_n)$. By Lemma 7, we get

$$T^n_{n-\lceil\frac{k}{2}\rceil+1}(\neg A_1, \neg A_2, \ldots, \neg A_n)$$

and adding this to itself using Lemmas 9 and 8, we obtain

$$T^{2n}_{2n-2\lceil\frac{k}{2}\rceil+2}(\neg A_1, \neg A_1, \neg A_2, \neg A_2, \ldots, \neg A_n, \neg A_n)\ .$$

Using Lemma 7 again yields

$$\neg T^{2n}_{2\lceil\frac{k}{2}\rceil-1}(A_1, A_1, A_2, A_2, \ldots, A_n, A_n)\ ,$$

and since $2\lceil\frac{k}{2}\rceil - 1 \leq k$, we get a contradiction by using Lemma 5. This argument can be formalized in PTK^* using cuts.

By the size and depth bounds for the lemmas used, the whole proof is of threshold depth 1 and of size polynomial in the size of the proof P. □

References

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