Exponential Incomparability of Tree-like and Ordered Resolution

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In [2], we have proved an exponential lower bound of the form $2^{\Omega(n \log n)}$ on the size of ordered resolution refutations of a certain set of clauses. Here we show that this set of clauses has quasi-polynomial size tree-like resolution refutations, thus giving an exponential separation of ordered from tree-like resolution. In particular, since tree-like refutations of minimal size are regular, it follows that regular resolution can have an exponential speed-up over ordered resolution. This answers a question that was left open in [2].

The exponential separation in the opposite direction follows from the work of [1]. They give an exponential lower bound of the form $2^{\Omega(n/\log n)}$ for tree-like resolutions of the pebbling clauses Peb_G associated to certain graphs G on n vertices, which have high pebbling number $\Omega(n/\log n)$. They also provide linear size, constant width dag-like resolution refutations of these clauses. It is easy to observe that these can even be obtained as ordered refutations.

Thus we have a strongly exponential separation of tree-like from ordered resolution in this direction also. A weakly exponential separation, with a lower bound of the form $2^{\Omega(n^\epsilon)}$, was already shown in [2].

The String-of-Pearls principle

From a bag of m pearls, which are colored red and blue, n pearls are chosen and placed on a string. The string-of-pearls principle $SP_{n,m}$ says that, if the first pearl is red and the last one is blue, then there must be a blue pearl next to a red pearl somewhere on the string. $SP_{n,m}$ is expressed by the following set of clauses in variables $p_{i,j}$ and r_j for $i \in [n]$ and $j \in [m]$, where $p_{i,j}$ means that pearl j is at position i on the string, and r_j means that pearl j is colored red:

$$\bigvee_{j=1}^{m} p_{i,j} \qquad i \in [n] \qquad (1)$$

$$\bar{p}_{i,k} \vee \bar{p}_{j,k} \qquad i,j \in [n],\ k \in [m],\ i \neq j \qquad (2)$$

$$\bar{p}_{i,j} \vee \bar{p}_{i,k} \qquad i \in [n],\ j,k \in [m],\ j \neq k \qquad (3)$$

$$p_{1,j} \to r_j \qquad j \in [m] \qquad (4)$$

$$p_{n,j} \to \bar{r}_j \qquad j \in [m] \qquad (5)$$

$$p_{i,j} \wedge r_j \wedge p_{(i+1),k} \to r_k \qquad 1 \leq i < n,\ j,k \in [m],\ j \neq k \qquad (6)$$

The string-of-pearls clauses $SP_{n,m}$ were introduced in [2], they are a modified and simplified version of the clauses related to the st-connectivity problem that were introduced by Clote and Setzer [3].

Theorem 1. *The clauses $SP_{n,m}$ have tree-like resolution refutations of size $nm^{O(\log n)}$.*

Proof. First we note that for $i < h < i' \in [n]$, the clauses

$$p_{i,j} \wedge r_j \wedge p_{i',j'} \to r_{j'} \qquad \text{for } j, j' \in [m]$$

each have a tree-like proof of size $O(m^2)$ from the $2m$ clauses

$$p_{i,j} \wedge r_j \wedge p_{h,k} \to r_k \quad \text{and} \quad p_{h,k} \wedge r_k \wedge p_{i',j'} \to r_{j'} \qquad \text{for } k \in [m]\ .$$

First, each pair of clauses is resolved with each other, eliminating the variable r_k, and then the resulting m clauses are resolved one by one with the axiom $\bigvee_{k=1}^{m} p_{h,k}$.

The set of clauses $p_{1,j} \wedge r_j \wedge p_{n,j'} \to r_{j'}$ for $j, j' \in [m]$ can be refuted in size $O(m^3)$ as follows: First they are resolved with the clauses $p_{1,j} \to r_j$, giving the clauses $p_{1,j} \wedge p_{n,j'} \to r_{j'}$. A proof as above of size $O(m^2)$ using the axiom $\bigvee_{j=1}^{m} p_{1,j}$ produces $p_{n,j'} \to r_{j'}$ for $j' \in [m]$. These are resolved with the clauses $p_{n,j'} \to \bar{r}_{j'}$, and the remaining unit clauses $p_{n,j'}$ can be resolved with the axiom $\bigvee_{j'=1}^{m} p_{n,j'}$.

To obtain the clauses $p_{1,j} \wedge r_j \wedge p_{n,j'} \to r_{j'}$, we form for each of them a $2m$-ary tree, in which each clause $p_{i,j} \wedge r_j \wedge p_{i',j'} \to r_{j'}$ is obtained from $2m$ clauses

$$p_{i,j} \wedge r_j \wedge p_{\lceil \frac{i+i'}{2} \rceil,k} \to r_k \ \text{ and } \ p_{\lceil \frac{i+i'}{2} \rceil,k} \wedge r_k \wedge p_{i',j'} \to r_{j'} \quad \text{for } k \in [m]\ .$$

as above. At the leaves, the axioms $p_{i,j} \wedge r_j \wedge p_{(i+1),j'} \to r_{j'}$ are used. Since the depth of the tree is $\lceil \log n \rceil$, it has $(2m)^{\lceil \log n \rceil + 1}$ many nodes, each corresponding to a subproof of size $O(m^2)$. As there are m^2 of these trees, the whole proof is of size at most $2n \cdot m^{\lceil \log n \rceil + 4}$. □

The clauses $SP_{n,m}$ are modified, giving clauses $SP'_{n,m}$ for which a lower bound on ordered resolutions can be proved, as follows: For $i \in [n]$ and $j \leq \frac{n}{4}$ define a certain value $f(i,j) \in [n]$. Then the clauses (4) and (6) for $1 \leq i < \frac{n}{2}$ are replaced by

$$p_{f(1,j),\ell} \wedge p_{1,j} \rightarrow r_j$$
$$p_{f(i+1,k),\ell} \wedge p_{i,j} \wedge r_j \wedge p_{(i+1),k} \rightarrow r_k$$

for every $\ell \in [m]$, and the clauses (5) and (6) for $\frac{n}{2} < i < n$ are replaced by

$$p_{f(n,j),\ell} \wedge p_{n,j} \rightarrow \bar{r}_j$$
$$p_{f(i,j),\ell} \wedge p_{i,j} \wedge r_j \wedge p_{(i+1),k} \rightarrow r_k$$

again for each $\ell \in [m]$. For details see [2], where the following theorem is proved:

Theorem 2. *The clauses $SP'_{n,m}$ for $m \geq \frac{9}{8}n$ require ordered resolution refutations of size $2^{\Omega(n \log n)}$.*

On the other hand, the original clauses (4), (5) and (6) can be derived from $SP'_{n,m}$ by small tree-like proofs, thus we obtain the following consequence of our proof above:

Corollary 3. *The clauses $SP'_{n,m}$ have tree-like resolution refutations of size $nm^{O(\log n)}$.*

Thus we have a strongly exponential separation between ordered and tree-like Resolution.

References

[1] E. Ben-Sasson, R. Impagliazzo, and A. Wigderson. Near-optimal separation of tree-like and general resolution. ECCC TR00-005, 2000.

[2] M. L. Bonet, J. L. Esteban, N. Galesi, and J. Johannsen. On the relative complexity of resolution restrictions and cutting planes proof systems. *SIAM Journal on Computing*, 30:1462–1484, 2000.

[3] P. Clote and A. Setzer. On PHP, st-connectivity and odd charged graphs. In P. Beame and S. R. Buss, editors, *Proof Complexity and Feasible Arithmetics*, pages 93–117. AMS DIMACS Series Vol. 39, 1998.