In previous work [1], the author has introduced the *dynamic ordinal* of a relativised theory T of Bounded Arithmetic, i.e., a theory with an additional uninterpreted predicate symbol α, as a measure of its strength. It bounds the length of induction for bounded universal ($\Pi^b_1(\alpha)$-) formulas provable in T. This notion is used to prove separations between the relativised theories.

In the paper under review, the notion is generalised: the i^{th} generalised dynamic ordinal of T measures the length of induction for $\Pi^b_i(\alpha)$-formulas available in T, i e., $DO_i(T)$ is the smallest set of functions such that T proves the schema

$$(\forall x < t)((\forall y < x)A(y) \to A(x)) \to (\forall x < t)A(x)$$

for all Π^b_i-formulas $A(x)$ if and only if t has a growth rate bounded by some function in $DO_i(T)$.

The $i{+}1^{\text{st}}$ dynamic ordinals of the theory defined by induction up to $|t|_m$, the m-fold iterated logarithm of a term, for $\Sigma^b_{m+i}(\alpha)$-formulas is computed as $2_m(O(|id|_{m+1})$, the m-fold iterated exponential of a constant multiple of the $(m+1)$-fold iterated logarithm. In particular, the i^{th} generalised dynamic ordinal of the theories $S^{i+1}_2(\alpha)$ and $T^i_2(\alpha)$ is the $2_2(O(||id||))$, i.e., the functions of growth exponential in a polylogarithm, which is just the maximal growth rate of terms in the language of Bounded Arithmetic. The i^{th} generalised dynamic ordinal of the theory $S^i_2(\alpha)$ is $2_1(O(||id||))$, and that of $R^{i+1}_2(\alpha)$ is $2_2(O(|id|_3))$. From these one can deduce most of the known separations between these theories, albeit not always with the best known quantifier complexity of the separating formula.

The upper bounds are shown by speed-up of induction technique common in proof theory, which allows to prove induction along longer orderings at the cost of increasing the quantifier complexity of the induction predicate.

The lower bounds are shown by a translation of proofs in the theories into propositional proof systems — certain formulations of constant-depth Frege proof systems — and then proving lower bounds on the height of proofs in these proof systems. These lower bound proofs utilize methods from Boolean circuit complexity theory introduced to the area by Krajíček [2], in particular Håstad's switching lemma and the substitution of Sipser functions for variables.

Finally the author notes that for certain theories T set $DO_i(T)$ coincides with the set of functions F such that the Σ^b_{i+2}-definable multivalued functions in T can be characterized by polynomial time computations with $\log f$ many calls to a Σ^b_{i+1} witness oracle. This is, however, so far only an observation, as no proof of this characterization using the generalised dynamic ordinal analysis is known.

References

[1] A. Beckmann, Dynamic ordinal analysis, Arch. Math. Logic **42** (2003), no. 4, 303–334. MR2018084 (2004m:03207)

[2] J. Krajíček, Lower bounds to the size of constant-depth propositional proofs, J. Symbolic Logic **59** (1994), no. 1, 73–86. MR1264964 (95k:03093)