Two Occurrences, Logarithmic Space and Perfect Matching

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Complexity of Satisfiability Problems

Some well-known facts:

The SAT problem for formulas in $\mathcal{F}$ is complete for the class **C**:

$\mathcal{F}$	**C**	Reference
CNF	**NP**	Cook 1971
Horn-CNF	**P**	Plaisted 1984
2-CNF	**NL**	Jones, Lien, Laaser 1976
2-$\oplus$CNF	**SL**	Jones, Lien, Laaser 1976

Question: What about **L** ?

Two Occurrences

Definition: CNF(2) is the set of CNF formulas with at most two occurrences of each variable.

SAT(2) is the SAT problem for formulas in CNF(2).

Known: SAT(2) $\in$ **LinTime**, SAT(3) is **NP**-complete.

Problem NAE-SAT:

Given: Formula F in CNF.

Question: Is there $\alpha \models F$ that also falsifies at least one literal in every clause in F ?

Theorem:

SAT(2) is complete for **L**.

NAE-SAT(2) is complete for **L**.

Tagged graphs

Definition: A tagged graph $G = (V, E, T)$ is an undirected multi-graph (V, E) with a subset $T \subseteq V$ of tagged vertices.

For $F \in \mathrm{CNF}(2)$, the tagged graph $G(F)$ is defined by:

- $G(F)$ has a vertex v_C for every clause C in F.
- If clauses C and D contain complementary literals $x, \bar{x}$, then there is an edge e_x between v_C and v_D.
- If C contains a pure literal, then $v_C \in T$.

Property: $F \in \mathrm{CNF}(2)$ is satisfiable iff $G(F)$ can be directed s.t. there is no untagged sink.

SAT(2) is in **L**

Lemma: $F \in \mathrm{CNF}(2)$ is satisfiable iff
(*) every component in $G(F)$ is tagged or contains a cycle.

Theorem: SAT(2) is in **L**.

Consider permutations of the darts D(G):

$$D(G) := \{\, (v,e) \,;\, v \text{ incident with } e \,\}$$

$$\rho_G := \prod_{v \in V} \big(\, (v,e_1) \,\ldots\, (v,e_k) \,\big)$$

$$\sigma_G := \prod_{e \in E} \big(\, (u,e)\,(v,e) \,\big)$$

Cook, McKenzie 1987: Product $\pi_G := \rho_G \circ \sigma_G$ can be computed in logarithmic space.

With the aid of π_G, property (*) can be decided in logarithmic space.

SAT(2) is hard for L

Problem TF:

Given: Undirected graph G.

Question: Does every component of G contain a cycle ?

Property: TF $\leq$ SAT(2)

Problem UFA:

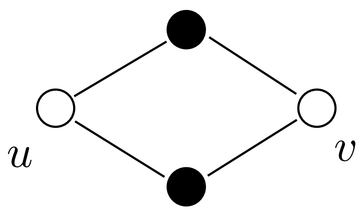
Given: Undirected forest G with exactly two trees, vertices u, v in G.

Question: Are u and v on the same tree ?

Fact: UFA is complete for L (Cook, McKenzie 1987)

Lemma: $\overline{\text{UFA}} \leq$ TF

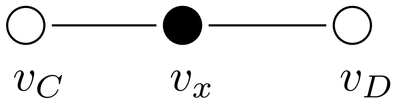
Reduction: Add to G two vertices connected to u, v:



Another tagged graph

For $F \in \mathrm{CNF}(2)$, the tagged graph $G'(F)$ is defined by:

- $G'(F)$ contains a vertex v_C for every clause C in F.
- Is $a \in C$ and $a \in D$, then there is an edge e_a between v_C und v_D.
- Is $x \in C$ und $\bar{x} \in D$, then there is a vertex v_x, and edges:



- If C contains a pure literal not occurring elsewhere, then $v_C \in T$.

NAE-SAT(2) **is in L**

Problem E2C:

Given: tagged graph $G = (V, E, T)$.

Question: Can E be coloured with 2 colours so that every untagged vertex has incident edges of both colours ?

Property: $F \in$ CNF(2) is in NAE-SAT iff $G'(F) \in$ E2C.

Lemma: $G \in$ E2C iff the following hold:

1. every untagged vertex has degree ≥ 2,
2. no untagged componente is a simple odd length cycle.

Corollary: NAE-SAT(2) is in **L**

NAE-SAT(2) is hard for **L**

Lemma: E2C $\leq$ NAE-SAT(2).

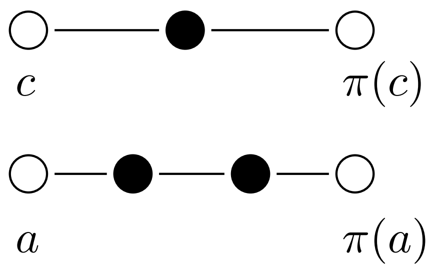
Problem DCA:

Given: Permutation $\pi \in S_n$, points $a, b \in [n]$.
Question: Are a and b on the same orbit of π?

Fact: DCA is complete for **L** (Cook, McKenzie 1987).

Lemma: DCA $\leq$ E2C.

Reduction: For every $c \in [n]$ two vertices, for a, b one more each, with edges:



$\oplus$SAT(2) and the class **SL**

Problem $\oplus$SAT:

Given: Formula F in CNF.

Question: Is there an assignment $\alpha \models F$, that sets an odd number of literals in every clause in F to true.

Problem UGAP:

Given: Undirected graph G, vertices a and b.

Question: Is there a path from a to b ?

Fact/Definition: **SL** is the class of problems $\leq_m^{\mathsf{L}}$-reducible to UGAP.

Theorem: $\oplus$SAT(2) is complete for **SL**.

$\oplus$SAT(2) is complete for **SL**

Definition: An edge coloring is admissible, if every untagged vertex has an odd number of incident edges colored red.

Property: $F \in$ CNF(2) is in $\oplus$SAT iff $G'(F)$ has an admssible coloring.

Lemma: G has an admissible coloring iff every untagged component of G is of even size.

Corollary: $\oplus$SAT(2) is in **SL**.

Lemma: $\oplus$SAT(2) is hard for **SL**.

Reduction UGAP $\leq \oplus$SAT(2):

Two copies of G, each vertex v connected to its copy v', two additional vertices a^* and b^* connected to a, a' and b, b' resp.

Exact Satisfiability

Problem XSAT:

Given: Formula F in CNF.

Question: Is there an assignment $\alpha \models F$, that sets exactly one literal in every clause in F to true.

Problem TPM:

Given: Tagged graph G.

Question: Is there a matching in G that matches every untagged vertex ?

PM: TPM for instances with $T = \emptyset$.

Property: $F \in \text{CNF}(2)$ is in XSAT iff $G'(F) \in \text{TPM}$.

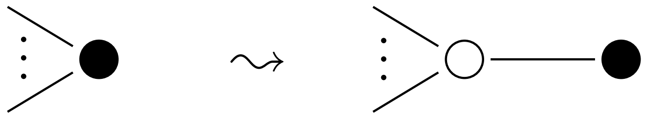
Corollary: $\text{XSAT}(2) \equiv \text{TPM}$.

XSAT(2) **is equivalent to** PM

Problem rTPM: TPM for instances, where every tagged vertex has degree 1.

Lemma: TPM $\equiv$ rTPM.

Reduktion $\leq$: Replace tagged vertices as follows:



Lemma: rTPM $\equiv$ PM.

Reduction $\leq$: Join all the tagged vertices, plus an additional one if $|V|$ is odd, in a large clique.

Korollar: XSAT(2) $\equiv$ PM

The Complexity of PM

Complexity of PM: hard for **NL**, in **RNC**, in non-uniform **SPL**, ...???

Definition: Class **PM**: Problems $\leq_m^{\mathsf{L}}$-reducible to PM.

Complete problems for the class **PM**:

- Matching: PM, TPM, rTPM, Maximum Matching.
- Exact Satisfiability: XSAT(2) in various forms.
- Flows: Maximum Flow with unary weights.
- Ramsey theory: Given G and n, does $G \not\rightarrow (K_{1,p}, K_{1,n})$?

A further problem in **PM**: UCC

Given: Undirected graph G.
Question: Does every component of G contain exactly one cycle ?

Some questions about the class **PM**

Question: Is **PM** closed under complement ?

Question: Is **PM** $= \leq_{FO}$ (PM) ?

Task: Give a machine characterization of the class **PM**.

Would probably be useful for the above questions !

Another approach: Study refutation systems for XUNSAT(2)

$\rightsquigarrow$ lower bounds to work towards showing **NL** $\subsetneq$ **PM**.