Calculus in Coinductive Form

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Introduction

Coinduction has only recently been recognized as a genuine logical principle. The idea in [EsPa98] is to put forward that coinduction is new only by name, while it had actually been around for a long time, concealed within the infinitistic methods of mathematical analysis. Roughly

$$\frac{induction}{arithmetic} \approx \frac{coinduction}{analysis}$$

The infinitary constructions in elementary calculus are coinductive, just like the infinitary constructions in elementary arithmetic are inductive.

The basic idea is to capture calculus via streams with the following definitions for the basic operations:

$$head(f) := f(0), \quad tail(f) := f', \quad a :: f := \lambda x.a + \int_0^x f(t)dt$$

What are saying the basic properties of Stream Algebras?

1. $head(a :: f) = a$. This corresponds to normalize the integral with respect to the interval of integration:
$$head(a :: f) = head(a + \int_0^x f(t)dt) = (a + \int_0^x f(t)dt)(0) = a + \int_0^0 f(t)dt = a$$
2. $tail(a :: f) = f$. This corresponds to normalize the integral with respect to the subintegral function.
$$tail(a :: f) = tail(a + \int_0^x f(t)dt) = \frac{d}{dx}(a + \int_0^x f(t)dt) = 0 + \frac{d}{dx}(\int_0^x f(t)dt) = f$$
3. $head(f) :: tail(f) = f$. This corresponds to the second fundamental theorem of calculus.
$$head(f) :: tail(f) = f(0) :: f' = f(0) + \int_0^x f'(t)dt = f$$

If we reapply the property 3 to the derivatives of f we obtain a stream which n-th entry is $f^{(n)}(0)$:

$$\begin{aligned} f &= f(0) :: f' \\ &= f(0) :: f'(0) :: f'' \\ &= f(0) :: f'(0) :: f''(0) :: f''' \\ &\vdots \\ &= f(0) :: f'(0) :: \ldots :: f^{(n)}(0) :: \ldots \end{aligned}$$

Moreover if we unfold the definition of :: , which amounts to iterated integration, we obtain the Taylor expansion for $f(x)$:

$$\begin{aligned}
f(x) &= f(0) :: f' = f(0) + \int_0^x f'(t)dt \\
&= f(0) + \int_0^x [f'(0) :: f''(t)]dt \\
&= f(0) + \int_0^x f'(0)dt + \int_0^x (\int_0^t f''(u)du)dt \\
&= f(0) + f'(0)x + \int_0^x \int_0^t f''(u)dudt \\
&= f(0) + f'(0)x + \int_0^x \int_0^t [f''(0) + \int_0^u f^{(3)}(v)dv]dudt \\
&= f(0) + f'(0)x + \int_0^x \int_0^t f''(0)dudt + \int_0^x \int_0^t \int_0^u f^{(3)}(v)dvdudt \\
&= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \int_0^x \int_0^t \int_0^u [f^{(3)}(0) + \int_0^v [f^{(4)}(0) + \ldots]]dvdudt \\
&\vdots \\
&= \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i
\end{aligned}$$

The idea of unfolding *ad infinitum* is formalized with a stream coalgebra.

1 Stream Algebras

The main tool are the Σ-*stream algebras* $\langle A, \mathsf{h}, \mathsf{t}, \mathsf{c}\rangle$ where A is a set and there is an isomorphism between $\Sigma \times A$ and A given by $\langle \mathsf{h}, \mathsf{t}\rangle : A \to \Sigma \times A$ with inverse $\mathsf{c} : \Sigma \times A \to A$.
In other words, Σ-stream algebras are just the final coalgebras of the functor $\Sigma \times (\cdot) : Set \to Set$.

1.1 Infinite Lists

The basic example of a stream algebra is the set of streams Σ^ω with the usual operations h (head), t (tail) and :: (cons). This stream algebra is the final coalgebra of the functor $\Sigma \times (\cdot)$.
From now on α, β denote infinite lists of the form $[\alpha_0, \alpha_1, \ldots], [\beta_0, \beta_1, \ldots]$.

1.2 Sequences

Another interesting example of stream algebra is $\langle \mathbb{Z}, O, \Delta, (\cdot) + \Sigma(\cdot)\rangle$, where

$$\begin{aligned}
O(\alpha) &= \alpha_0 \\
\Delta(\alpha) &= [\alpha_1 - \alpha_0, \alpha_2 - \alpha_1, \ldots, \alpha_{n+1} - \alpha_n, \ldots] \\
a + \Sigma\beta &= [a, a + \beta_0, a + \beta_0 + \beta_1, \ldots\ldots]
\end{aligned}$$

This stream algebra is isomorphic with the final coalgebra $\mathbb{Z}^\omega$ via $\tau, \widetilde{\tau} : \mathbb{Z}^\omega \to \mathbb{Z}^\omega$ with entries: $\tau(\alpha)_n = \sum_{i=0}^{n} \binom{-n}{i} \alpha_i$ and $\widetilde{\tau}(\alpha)_n = \sum_{i=0}^{n} \binom{n}{i} \alpha_i$.

1.3 Analytic Functions

We formalize the basic idea to capture calculus with a stream algebra over the set $\mathbb{A}$ of functions analytic at 0, via the following operations:

$$\begin{aligned}
O &: \mathbb{A} \to \mathbb{R} & O(f) &= f(0) \\
D &: \mathbb{A} \to \mathbb{A} & D(f) &= f' \\
C &: \mathbb{R} \times \mathbb{A} \to \mathbb{A} & C(a, f) &= a + \int_0^x f(t)dt
\end{aligned}$$

2 Solving Equations

From another point of view $C(a, f) := a :: f$ is the unique solution of the differential equation $g' = f$ with the initial value $g(0) = a$ and the derivation of the Taylor series by unfolding leads to the usual power series method for solving differential equations.
To every initial value problem of the form:

$$y^{(n)} = F(x, y), \;\; y(0) = a_0, \ldots, y^{(n-1)}(0) = a_{n-1} \tag{1}$$

corresponds a stream equation

$$y = a_0 :: a_0 :: \ldots :: a_{n-1} :: F(x, y) \tag{2}$$

For example to the differential equation $y^(4) = y$ with $y(0) = 0 = y''(0), y'(0) = 1, y'''(0) = -1$ corresponds the stream equation $y = 0 :: 1 :: 0 :: -1 :: y$ and in this case the solution is $y = \sin x$. Elementary functions arise as solutions of equations,e.g $\exp = 1 :: \exp$ corresponds to the exponential function and $\cosh = 1 :: 0 :: \cosh$ to the hiperbolic cosine.

To solve this kind of equations we have to represent real numbers as well as variables in $\mathbb{A}$.

2.1 Lifting the structure

To lift the real numbers into $\mathbb{A}$ we first define $\widehat{0}$ as the unique solution of the equation $\widehat{0} = 0 :: \widehat{0}$. Thus we identify $a \in \mathbb{R}$ with the constant function $\widehat{a} = a :: \widehat{0}$.
The variable x is now defined as $0 :: 1$, $x^2 := 0 :: 0 :: 2$, $x^3 := 0 :: 0 :: 0 :: 6$, etc.

To explain the representation of x^n we need to define multiplication in $\mathbb{A}$. Moreover $\mathbb{A}$ has a ring structure, the sum is the usual pointwise sum:

$$O(f + g) = O(f) + O(g), \;\; D(f + g) = Df + Dg$$

and the product is given by:

$$O(f \cdot g) = O(f) \cdot O(g), \;\; D(f \cdot g) = Df \cdot g + f \cdot Dg$$

2.2 Using Taylor Series

Definition 2.1 Let $\mathbb{R}^{<\omega} = \{\alpha \in \mathbb{R}^{\omega} | \sum_{i=0}^{\infty} \frac{\alpha_i}{i!} x^i < \infty, \text{ for some } x > 0\}$, i.e.,$\mathbb{R}^{<\omega}$ is the set of streams of Taylor coefficients.
Let $\mathsf{T} : \mathbb{A} \rightarrow \mathbb{R}^{<\omega}$ be defined by

$$\mathsf{T}(f) = [f(0), f'(0), f''(0), \ldots, f^{(n)}(0), \ldots]$$

T is the *Taylor transformation.*

It is obvious that T is bijective and the inverse is $\widetilde{\mathsf{T}} : \mathbb{R}^{<\omega} \rightarrow \mathbb{A}$ with $\widetilde{\mathsf{T}}(\alpha)(x) = \sum_{i=0}^{\infty} \frac{\alpha_i}{i!} x^i$.
Moreover we have the following

Proposition 2.1 $\langle \mathbb{A}, O, D, C\rangle \cong \langle \mathbb{R}^{<\omega}, h, t, ::\rangle$

Proof:

Choose $\widetilde{\mathsf{T}}$ and T as the functions between the carriers $\dashv$

Therefore we can forget about analytic functions, its values at 0, and its derivatives and work with the streams of coefficients of its Taylor Series.

Now that we have T available, we can solve stream equations like (2) using the additivity of T as well as the property $T(a :: f) = a :: T(f)$ (i.e. $T(C(a, f)) = a :: T(f)$).

3 Laplace Transform

3.1 Rings of Streams

Now consider the algebra $\langle \mathbb{R}^{\omega}, h, t^{\mathbb{N}}, c^{\mathbb{N}} \rangle$, with

$$\begin{aligned} t^{\mathbb{N}}(\alpha) &= [\alpha_1, 2\alpha_2, 3\alpha_3, \ldots, n\alpha_n, \ldots] \\ c^{\mathbb{N}}(a, \beta) &= [a, \beta_0, \tfrac{\beta_1}{2}, \tfrac{\beta_2}{3} \ldots, \tfrac{\beta_n}{n+1}, \ldots] \end{aligned}$$

Proposition 3.1 $\langle \mathbb{R}^{\omega}, h, t, :: \rangle \cong \langle \mathbb{R}^{\omega}, h, t^{\mathbb{N}}, c^{\mathbb{N}} \rangle$

Proof:

Use

$$\begin{array}{ll} g : \mathbb{R}^{\omega} \to \mathbb{R}^{\omega}, & g(\alpha)_n = n!\alpha_n \\ \widetilde{g} : \mathbb{R}^{\omega} \to \mathbb{R}^{\omega}, & \widetilde{g}(\alpha)_n = \frac{\alpha_n}{n!} \end{array}$$

$\dashv$

Take $G = \widetilde{\mathsf{T}} \circ g$, then G asigns to every stream α , its generating function $f_{\alpha}(x) = \sum_{i=0}^{\infty} \alpha_i x^i$.

With respect to G now $t^{\mathbb{N}}$ represents derivation and $c^{\mathbb{N}}$ represents integration:

$$\tfrac{d}{dx} G(\alpha)(x) = \textstyle\sum_{i=1}^{\infty} i\alpha_i x^{i-1} = G(t^{\mathbb{N}}(\alpha))$$

$$\textstyle\int G(\beta)(x)dx = \int \sum_{i=0}^{\infty} \beta_i x^i dx = a + \sum_{i=0}^{\infty} \frac{\beta_i}{i+1} x^{i+1} = G(c^{\mathbb{N}}(a, \beta))$$

where a is the integration constant.

Next we induce in $\mathbb{R}^{<\omega}$ two ring structures, in both the sum $+$ is just the componentwise sum. Recall that $\mathbb{R}^{<\omega} = \{\mathsf{T}(f) | f \in \mathbb{A}\}$. For $\varphi = \mathsf{T}(f), \gamma = \mathsf{T}(h)$ we define $\varphi \cdot \gamma := g \circ \mathsf{T}(fh)$, with entries:

$$(\varphi \cdot \gamma)_n = \sum_{i=0}^{n} \binom{n}{k} \varphi_i \gamma_{n-i}$$

Now consider $\widetilde{G} = \widetilde{g} \circ \mathsf{T}$. We have $\mathbb{R}^{<\omega} = \{\widetilde{G}(f) | f \in \mathbb{A}\}$ and for $\alpha = \widetilde{G}(f), \beta = \widetilde{G}(h)$ we define $\alpha \star \beta := \widetilde{G}(fh)$, with entries:

$$(\alpha \star \beta)_n = \sum_{i=0}^{n} \alpha_i \beta_{n-i}$$

Proposition 3.2 $\langle \mathbb{R}^{<\omega}, +, \cdot \rangle \cong \langle \mathbb{R}^{<\omega}, +, \star \rangle$

Proof:

The function g is a ring isomorphism. $\dashv$

In particular g satisfies: $g(\alpha \star \beta) = g(\alpha) \cdot g(\beta)$

The importance of the ring $\langle \mathbb{R}^{<\omega}, +, \star \rangle$ is that integration becomes multiplication by x in this ring. We have for every $f \in \mathbb{A}$:

$$\mathsf{T}\,(\int_0^x f(t)dt) = x \star \mathsf{T}\,(f)$$

Moreover, the ring $\langle \mathbb{R}^{<\omega}, +, \star \rangle$ is an integer domain, therefore can be extended into a field, and in this field calculus becomes algebra. This is the idea of Mikusiński's Operator Calculus.

3.2 Laplace Transform of Taylor Coefficients

The crucial point about Laplace Transform is that it is not an isomorphism but a proper embedding of the convolution ring of real analytic functions into an ideal, within the multiplicative ring of holomorphic complex functions.

First we embed $\mathbb{R}^{\omega}$ into $\mathbb{R}^{\omega}_{0} = \{a \in \mathbb{R}^{\omega} | \alpha_0 = 0\}$ via the following stream algebra:

$$\chi : \mathbb{R}^{\omega} \to \mathbb{R} \qquad \alpha \mapsto \alpha_1$$

$$\vartheta : \mathbb{R}^{\omega} \to \mathbb{R}^{\omega} \qquad \alpha \mapsto [\alpha_0, \alpha_2, \tfrac{\alpha_3}{2}, \tfrac{\alpha_4}{3}, \ldots, \tfrac{\alpha_{n+1}}{n}, \ldots]$$

$$\varsigma : \mathbb{R} \times \mathbb{R}^{\omega} \to \mathbb{R}^{\omega} \quad \langle a, \beta\rangle \mapsto [\beta_0, a, \beta_1, 2\beta_2, 3\beta_3, \ldots, (n-1)\beta_{n-1}]$$

Proposition 3.3 $\langle \mathbb{R}^{\omega}, h, t, ::\rangle$ embeds into $\langle \mathbb{R}^{\omega}_{0}, \chi, \vartheta, \varsigma\rangle$

Proof:

Consider the function $\ell : \mathbb{R}^{\omega} \to \mathbb{R}^{\omega}$ with entries:

$$\ell(\alpha)_n = \begin{cases} 0 & \text{if } n = 0 \\ n!\alpha_{n-1} & \text{if } n > 0 \end{cases}$$

We have that $\ell[\mathbb{R}^{\omega}] = \mathbb{R}^{\omega}_{0}$ and $\langle \mathbb{R}^{\omega}_{0}, \chi \restriction_{\mathbb{R}^{\omega}_{0}}, \vartheta \restriction_{\mathbb{R}^{\omega}_{0}}, \varsigma \restriction_{\mathbb{R}^{\omega}_{0}}\rangle \cong \langle \mathbb{R}^{\omega}, h, t, ::\rangle$. ⊣

ℓ preserves the zero and the sum:

$$\ell(\alpha + \beta) = \ell(\alpha) + \ell(\beta)$$

But not the product:

$$\ell(\alpha \star \beta) \neq \ell(\alpha) \cdot \ell(\beta)$$

If we take $\alpha = \beta = 1 = [1, 0, 0, \ldots]$ then $1 \star 1 = 1$ and $\ell(1) = x$. On the other hand $\ell(1) \cdot \ell(1) = x \cdot x = x^2 \neq x$.

About this Pavlović has answered me:

the equation certainly holds (indeed, the isomorphism of the two ring structures is the essence of the laplace transform) — but not as it stands in the paper: not for the series convolution, but rather for the functional one (which is exactly what is needed for the sequel, to get $\mathcal{L}(f \star g) = \ell(T(f) \star T(g))$), as derived in most distribution theory books (eg Mikusinski).

3.3 Laplace Transform of Analytic Functions

Definition 3.1 Let f be complex function holomorphic at ∞, i.e. there is a Laurent expansion $f(z) = \sum_{i=0}^{\infty} \frac{\alpha_i}{z_i}$. We say that f is *coanalytic* if all coefficients α_i are real.
Let $\mathbb{H}$ be the set of coanalytic functions.

Lemma 3.1 $f(z) \in \mathbb{H}$ if and only if $f(\frac{1}{x}) \in \mathbb{A}$. Conversely every real function $g(x)$ gives rise to a coanalytic function $g(\frac{1}{z})$

Therefore we have a one-to-one correspondence $\hbar : \mathbb{H} \to \mathbb{A}$. Extending it along T yields the bijection $\mathsf{T} \circ \hbar =: \frac{1}{\mathsf{T}} : \mathbb{H} \to \mathbb{R}^{<\omega}$, which assigns to each coanalytic function $f(z)$ the Taylor coefficients of $f(\frac{1}{x})$. And its inverse is: $\widetilde{\frac{1}{\mathsf{T}}}(\alpha) = \sum_{i=0}^{\infty} \frac{\alpha_i}{i!s^i}$, where s is a complex variable.

Finally we get a correspondence between the function ℓ and the Laplace Transform $\mathcal{L}$

Proposition 3.4 Let $\alpha = T(f), \beta = \frac{1}{\mathsf{T}}(g)$. Then

$$\mathcal{L}(f) = g \Leftrightarrow \ell(\alpha) = \beta$$

The obvious consequence is that for every $f \in \mathbb{A}$ holds:

$$\mathcal{L}(f) = \widetilde{\frac{1}{\mathsf{T}}} \circ \ell \circ \mathsf{T}\ (f)$$

Corollary 3.1 Let $\mathbb{H}_\infty = \{g \in \mathbb{H} |\ g \text{ vanishes at } \infty\}$. Then $\mathcal{L} : \mathbb{A} \to \mathbb{H}_\infty$ is a bijection.

Moreover

Corollary 3.2 $\mathcal{L} : \mathbb{A} \to \mathbb{H}_\infty$ is the only continuous linear operator satisfying

$$\mathcal{L}(x^n) = \frac{n!}{s^{n+1}}$$

References

[EsPa98] M.H. Escardó, D. Pavlović. Calculus in Coinductive Form. Proceedings of the 13th Annual IEEE Symposium on Logic in Computer Science. Indiana, USA, June 1998.

[PaPr99] D. Pavlović, V. Pratt. On Coalgebra of Real Numbers. Electronic Notes in Theoretical Computer Science 19 (1999)

[Bra66] L. Brand. Differential and Difference Equations. J. Wiley 1966.

A Calculus

Theorem A.1 [First Fundamental Theorem of Calculus] Let f be a continuous function on $[a, b]$ and $F(x) = \int_a^x f(t)dt$. Then F is differentiable and F'(x) = f(x), i.e.

$$\frac{d}{dx} \int_a^x f(t)dt = f$$

Theorem A.2 [Second Fundamental Theorem of Calculus] Let f be a continuous function on $[a, b]$ and F any antiderivative of f (i.e. $F' = f$) then

$$\int_a^b f(t)dt = F(b) - F(a)$$

In particular, if f is continuous at 0, then $f(x) = f(0) + \int_0^x f'(t)dt$.

Definition A.1 The power series

$$\sum_{i=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the *Taylor Series* of f *about* x_0. From now on we consider only Taylor series about 0, $f(x) = \sum_{i=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ (also called McLaurin Series).

Definition A.2 The funcion $f : \mathbb{R} \to \mathbb{R}$ is *analytic* at x_0 , if it is continuous and has derivatives of all orders for $|x - x_0| < \rho$, for some $\rho > 0$.
Alternatively, f is analytic at x_0, if it has a Taylor series expansion

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

about x_0.
Let $\mathbb{A} = \{f : \mathbb{R} \to \mathbb{R} | f \text{ is analytic at } 0\}$

Theorem A.3 [Basic Properties of Power Series] The following are valid for an arbitrary power series and for $|x| < \rho$, for some $\rho > 0$. In particular are valid for Taylor series.

- Product of Series: If $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $h(x) = \sum_{i=0}^{\infty} b_i x^i$. Then

$$f(x)h(x) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} a_j b_{k-j} x^k$$

- Differentiation of a Series. Suppose that $f(x) = \sum_{i=0}^{\infty} a_i x^i$. Then f is a differentiable function and

$$f'(x) = \sum_{i=1}^{\infty} \frac{d}{dx} a_i x^i = \sum_{i=1}^{\infty} i a_i x^{i-1}$$

- Integration of a Series: Suppose that $f(x) = \sum_{i=0}^{\infty} a_i x^i$. Then f is an integrable function and

$$\int_0^x f(t)dt = \sum_{i=0}^{\infty} \int_0^x a_i t^i dt = \sum_{i=0}^{\infty} \frac{a_i}{i+1} x^{i+1}$$

B The Laplace Transform

The Laplace transform is an integral transform which has the property of translating certain complicated operations (e.g. the differentiation of a function or the convolution of two functions), into simple algebraic operations in the image (Laplace) space. It can therefore be used to transform certain types of functional equations into algebraic equations.

Definition B.1 Let $f(t)$ be a function defined on $[0, \infty)$. The Laplace transform of $f(t)$ is a new function $\mathcal{L}(f)$ defined as

$$\mathcal{L}(f)(s) = \int_0^{\infty} e^{-st} f(t)dt$$

where s is a complex variable.

Proposition B.1 The Laplace transform $\mathcal{L}(f)$ has the following properties:

1. $\mathcal{L}$ is linear and continuous, i.e.

$$\mathcal{L}(\sum_{i=0}^{\infty} \alpha_i x^i) = \sum_{i=0}^{\infty} \alpha_i \mathcal{L}(x^i)$$

2. $\mathcal{L}(x^i) = \frac{i!}{s^{i+1}}$

C Mikusiński's Operator Calculus

Mikusiński has developed an operational calculus that casts a new light on Laplace transforms methods,in effect freeing them from considerations of convergence introduceb by the improper integral $\int_0^{\infty} e^{-st} f(t)dt$ and bringing the essential theory into the realm of algebra by means of a commutative ring in which the elements are the class of continuous real or complex-valued functions over $[0, \infty)$ and the operations are pointwise sum and convolution product:

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (f \star g)(x) &= \int_0^x f(t)g(x-t)dt \end{aligned}$$

This ring is indeed an integer domain (Titchmarsh's Theorem) but in the field of fractions, the quotient $\frac{f}{g}$ may not be a function at all, but is called *operator* by Mikusiński. An important operator is $s := \frac{1}{h}$, where h is the Heaviside unit function

$$h(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

If a is a number, then the operator $\frac{a}{h}$ behaves in $+, \star$ just as numbers in $+, \cdot$ and is denoted by a. Now we can state the basic formula of operator calculus: if f' is continuous for all $t \geq 0$ then

$$f' = sf - f(0)$$

See [Bra66].