

Minimal Stable Sets in Tournaments

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We propose a systematic methodology for defining tournament solutions as extensions of maximality. The central concepts of this methodology are maximal qualified subsets and minimal stable sets. We thus obtain an infinite hierarchy of tournament solutions, which encompasses the top cycle, the uncovered set, the Banks set, the minimal covering set, the tournament equilibrium set, the Copeland set, and the bipartisan set. Moreover, the hierarchy includes a new tournament solution, the *minimal extending set*, which is conjectured to refine both the minimal covering set and the Banks set.

1 Introduction

Given a finite set of alternatives and choices between all pairs of alternatives, how to choose from the entire set in a way that is faithful to the pairwise comparisons? This simple, yet captivating, problem is studied in the literature on tournament solutions. A tournament solution thus seeks to identify the “best” elements according to some binary dominance relation, which is usually assumed to be asymmetric and complete. As the ordinary notion of maximality may return no elements due to cyclical dominations, numerous alternative solution concepts have been devised and axiomatized (see, e.g., Moulin, 1986; Laslier, 1997). In social choice theory, the dominance relation is typically defined via pairwise majority voting, and many well-known tournament solutions constitute attractive social choice correspondences.

In this paper, we approach the tournament choice problem using a methodology consisting of two layers: *qualified subsets* and *stable sets*. Our framework captures most known tournament solutions (notable omissions are the Slater set and the Markov set).

The point of departure for our methodology is to collect the maximal elements of so-called qualified subsets because the set of *all* alternatives may not admit a maximal element. Examples of families of qualified subsets are *subsets with at most two elements*, *maximal subsets that admit a maximal element*, and *maximal transitive subsets*. This yields the set of all alternatives except the minimum, the uncovered set (Fishburn,

1977; Miller, 1980), and the Banks set (Banks, 1985), respectively. It is shown that the Banks set is the finest tournament solution definable via qualified subsets and can be axiomatically characterized using a single consistency condition and minimality. Generalizing an idea by Dutta (1988), we then propose a general method for refining any suitable solution concept S by defining minimal sets that satisfy certain conditions of internal and external stability with respect to S . When applying this method to the concepts mentioned above, we obtain the *minimal dominating set*, better known as the *top cycle* (Good, 1971; Smith, 1973), the *minimal covering set* (Dutta, 1988), and a new tournament solution that we call the *minimal extending set (ME)*.

Assuming that every tournament admits a *unique* minimal stable set with respect to the Banks set, which we conjecture to be the case, ME is contained in both the minimal covering set and the Banks set and satisfies a number of desirable properties. In this respect, ME bears some resemblance to Schwartz’s tournament equilibrium set TEQ (Schwartz, 1990), whose characteristics depend on a similar conjecture made by Schwartz (Laffond et al., 1993a; Houy, 2009). We show that Schwartz’s conjecture is stronger than ours and has a number of interesting consequences such as that TEQ can also be represented as a minimal stable set and is strictly contained in ME . As a result we obtain three new axiomatic characterizations of TEQ .

We also briefly discuss a quantitative version of our framework, which considers qualified subsets that are maximal in terms of cardinality rather than set inclusion. We thus obtain the Copeland set (Copeland, 1951) and—using a slightly modified definition of stability—the bipartisan set (Laffond et al., 1993b).

2 Preliminaries

The core of the problem studied in the literature on tournament solutions is how to extend choices in sets consisting of only two elements to larger sets. Thus, our primary objects of study will be functions that select one alternative from any pair of alternatives. Any such function can be conveniently represented by a tournament, i.e., a binary relation on the entire set of alternatives. Tournament solutions then advocate different views on how to choose from arbitrary subsets of alternatives based on these pairwise comparisons (see, e.g., Laslier, 1997, for an excellent overview of tournament solutions and their properties).

2.1 Tournaments

Let X be a universe of alternatives. For notational convenience we assume that $\mathbb{N} \subseteq X$. The set of all *finite* subsets of set X will be denoted by $\mathcal{F}_0(X)$ whereas the set of all *non-empty* finite subsets of X will be denoted by $\mathcal{F}(X)$. A (finite) *tournament* T is a pair (A, \succ) , where $A \in \mathcal{F}_0(X)$ and \succ is an asymmetric and complete (and thus irreflexive) binary relation on X , usually referred to as the *dominance relation*.¹ Intuitively, $a \succ b$

¹This definition slightly diverges from the common graph-theoretic definition where \succ is defined on A rather than X . However, it facilitates the sound definition of tournament functions (such as

signifies that alternative a is preferable to b . The reflexive closure of the dominance relation will be denoted by \succeq . The dominance relation can be extended to sets of alternatives by writing $A \succ B$ when $a \succ b$ for all $a \in A$ and $b \in B$. We further write $\mathcal{D}(X) \subseteq X \times X$ for the set of all dominance relations on X and $\mathcal{T}(X) = \mathcal{F}(X) \times \mathcal{D}(X)$ for the set of all tournaments on X .

For a relation R , we define R^k recursively by letting $R^1 = R$ and $R^{k+1} = R^k \cup \{(a, b) \mid (a, c) \in R^k \text{ and } (c, b) \in R \text{ for some } c\}$. The transitive closure of R is defined as $R^* = \bigcup_{k \in \mathbb{N}} R^k$. For a set B , a relation R , and an element a , we denote by $D_{B,R}(a)$ the *dominion* of a in B , i.e.,

$$D_{B,R}(a) = \{b \in B \mid a R b\},$$

and by $\overline{D}_{B,R}(a)$ the *dominators* of a in B , i.e.,

$$\overline{D}_{B,R}(a) = \{b \in B \mid b R a\}.$$

Whenever the tournament (A, \succ) is known from the context and R is the dominance relation \succ or B is the set of all alternatives A , the respective subscript will be omitted to improve readability.

The *order* of a tournament $T = (A, \succ)$ refers to its number of alternatives $|A|$. A tournament $T = (A, \succ)$ is called *regular* if $|D(a)| = |D(b)|$ for all $a, b \in A$. Only tournaments of odd order can be regular and regular tournaments of arbitrary odd order are easily constructed.

Let $T = (A, \succ)$ and $T' = (A', \succ')$ be two tournaments. A *tournament isomorphism* of T and T' is a bijective mapping $\pi : A \rightarrow A'$ such that $a \succ b$ if and only if $\pi(a) \succ' \pi(b)$. A tournament $T = (A, \succ)$ is called *homogeneous* (or vertex-transitive) if for every pair of alternatives $a, b \in A$ there is a tournament automorphism $\pi : A \rightarrow A$ of T such that $\pi(a) = b$.

2.2 Components and Decompositions

An important structural concept in the context of tournaments are *components*. A component is a subset of alternatives that bear the same relationship to all alternatives not in the set.

Definition 1. Let $T = (A, \succ)$ be a tournament. A non-empty subset B of A is a *component* of T if for all $a \in A \setminus B$ either $B \succ a$ or $a \succ B$. A *decomposition* of T is a set of components $\{B_1, \dots, B_k\}$ of T such that $A = \bigsqcup_{i=1}^k B_i$.

The *null decomposition* of a tournament $T = (A, \succ)$ is $\{A\}$; the *trivial decomposition* consists of all singletons of A . Any other decomposition is called *proper*. A tournament is said to be *decomposable* if it admits a proper decomposition. Given a particular decomposition, the *summary* of a tournament is defined as the tournament on the individual components rather than the alternatives.

tournament solutions or concepts of qualified subsets).

Definition 2. Let $T = (A, \succ)$ be a tournament and $\tilde{B} = \{B_1, \dots, B_k\}$ a decomposition of T . The *summary* $\tilde{T} = (\tilde{B}, \tilde{\succ})$ of T is the tournament such that for any $i, j \in \{1, \dots, k\}$ with $i \neq j$, $B_i \tilde{\succ} B_j$ if and only if $B_i \succ B_j$.

Conversely, a new tournament can be constructed by replacing each alternative with a component.

Definition 3. Let $B_1, \dots, B_k \subseteq X$ be disjoint sets and $\tilde{T} = (\{1, \dots, k\}, \tilde{\succ})$, $T_1 = (B_1, \succ_1), \dots, T_k = (B_k, \succ_k)$ tournaments. The *product* of T_1, \dots, T_k with respect to \tilde{T} , denoted by $\Pi(\tilde{T}, T_1, \dots, T_k)$, is the tournament (A, \succ) such that $A = \bigsqcup_{i=1}^k B_i$ and for all $b_i \in B_i, b_j \in B_j$, $b_i \succ b_j$ if only if $i = j$ and $b_i \succ_i b_j$, or $i \neq j$ and $i \tilde{\succ} j$.

Components can also be used to simplify the graphical representation of tournaments. We will denote components by gray ellipses. Furthermore, omitted edges in figures that depict tournaments are assumed to point downwards by convention (see, e.g., Figure 1 on page 17).

2.3 Tournament Functions

A central aspect of this paper will be functions that, for a given tournament, yield one or more subsets of alternatives. We will therefore define the notion of a *tournament function*. A function on tournaments is a tournament function if it is independent of outside alternatives and stable with respect to tournament isomorphisms.

Definition 4. Let $Z \in \{\mathcal{F}_0(X), \mathcal{F}(X), \mathcal{F}(\mathcal{F}(X))\}$. A function $f : \mathcal{T}(X) \rightarrow Z$ is a *tournament function* if

- (i) $f(T) = f(T')$ for all tournaments $T = (A, \succ)$ and $T' = (A, \succ')$ such that $\succ|_A = \succ'|_A$, and
- (ii) $f((\pi(A), \succ')) = \pi(f(A, \succ))$ for all tournaments (A, \succ) , (A', \succ') , and tournament isomorphisms $\pi : A \rightarrow A'$ of (A, \succ) and (A', \succ') .

For a given set $B \in \mathcal{F}(X)$ and tournament function f , we overload f by also writing $f(B)$, provided the dominance relation is known from the context. For two tournament functions f and f' , we write $f' \subseteq f$ if $f'(T) \subseteq f(T)$ for all tournaments T .

2.4 Tournament Solutions

The first tournament function we consider is the *maximum* function $\max_{\prec} : \mathcal{T}(X) \rightarrow \mathcal{F}_0(X)$, which is defined as

$$\max_{\prec}((A, \succ)) = \{a \in A \mid a \succ A \setminus \{a\}\}.$$

Due to the asymmetry of the dominance relation, the maximum function returns at most one alternative in any tournament. Moreover, maximal—i.e., undominated—and maximum elements coincide. In social choice theory, the maximum of a majority tournament

is commonly referred to as the *Condorcet winner*. Obviously, the dominance relation may contain cycles and thus fail to have a maximal element. For this reason, a variety of concepts have been suggested to take over the maximum's role of singling out the “best” alternatives of a tournament. Formally, a *tournament solution* S is defined as a function that associates with each tournament $T = (A, \succ)$ a non-empty subset $S(T)$ of A .

Definition 5. A *tournament solution* S is a tournament function $S : \mathcal{T}(X) \rightarrow \mathcal{F}(X)$ such that $\max_{\succ}(T) \subseteq S(T) \subseteq A$ for all tournaments $T = (A, \succ)$.²

The set $S(T)$ returned by a tournament solution for a given tournament T is called the *choice set* of T whereas $A \setminus S(T)$ consists of the *unchosen alternatives*. Since tournament solutions always yield non-empty choice sets, they have to select all alternatives in homogeneous tournaments. If $S' \subseteq S$ for two tournament solutions S and S' , we say that S' is a *refinement* of S or that S' is *finer* than S .

2.5 Properties of Tournament Solutions

The literature on tournament solutions has identified a number of desirable properties for these concepts. In this section, we will define six of the most typical properties.³ In a more general context, Moulin (1988) distinguishes between *monotonicity* and *independence* conditions, where a monotonicity condition describes the positive association of the solution with some parameter and an independence condition characterizes the invariance of the solution under the modification of some parameter.

In the context of tournament solutions, we will further distinguish between properties that are defined in terms of the dominance relation and properties defined in terms of the set inclusion relation. With respect to the former, we consider *monotonicity* and *independence of unchosen alternatives*. A tournament solution is monotonic if a chosen alternative remains in the choice set when extending its dominion and leaving everything else unchanged.

Definition 6. A tournament solution S satisfies *monotonicity* (MON) if $a \in S(T)$ implies $a \in S(T')$ for all tournaments $T = (A, \succ)$, $T' = (A, \succ')$, and $a \in A$ such that $\succ|_{A \setminus \{a\}} = \succ'|_{A \setminus \{a\}}$ and $D_{\succ}(a) \subseteq D_{\succ'}(a)$.

A solution is independent of unchosen alternatives if the choice set is invariant under any modification of the dominance relation between unchosen alternatives.

Definition 7. A tournament solution S is *independent of unchosen alternatives* (IUA) if $S(T) = S(T')$ for all tournaments $T = (A, \succ)$ and $T' = (A, \succ')$ such that $\succ|_{S(T) \cup \{a\}} = \succ'|_{S(T) \cup \{a\}}$ for all $a \in A$.

²Laslier (1997) is slightly more stringent here as he requires the maximum be the only element in $S(T)$ whenever it exists.

³Our terminology slightly differs from the one by Laslier (1997) and others. *Independence of unchosen alternatives* is also called *independence of the losers* or *independence of non-winners*. The *weak superset property* has been referred to as ϵ^+ or the *Aizerman property*.

With respect to set inclusion, we consider a monotonicity property to be called the *weak superset property* and an independence property known as the *strong superset property*. A tournament solution satisfies the weak superset property if an unchosen alternative remains unchosen when removing other unchosen alternatives.

Definition 8. A tournament solution S satisfies the *weak superset property* (WSP) if $S(B) \subseteq S(A)$ for all tournaments $T = (A, \succ)$ and $B \subseteq A$ such that $S(A) \subseteq B$.

The strong superset property states that a choice set is invariant under the removal of alternatives not in the choice set.

Definition 9. A tournament solution S satisfies the *strong superset property* (SSP) if $S(B) = S(A)$ for all tournaments $T = (A, \succ)$ and $B \subseteq A$ such that $S(A) \subseteq B$.

The difference between WSP and SSP is precisely another independence condition called *idempotency*. A solution is *idempotent* (IDE) if the choice set is invariant under repeated application of the solution concept, i.e., $S(S(T)) = S(T)$ for all tournaments T . When S is not idempotent, we define $S^k(T) = S(S^{k-1}(T))$ inductively by letting $S^1(T) = S(T)$ and $S^\infty(T) = \bigcap_{k \in \mathbb{N}} S^k(T)$.

The four properties defined above (MON, IUA, WSP, and SSP) will be called *basic* properties of tournament solutions. The conjunction of MON and SSP implies IUA. It is therefore sufficient to show MON and SSP in order to prove that a tournament solution satisfies all four basic properties.

Two further properties considered in this paper are *composition-consistency* and *irregularity*. A tournament solution is composition-consistent if it chooses the “best” alternatives from the “best” components.

Definition 10. A tournament solution S is *composition-consistent* (COM) if for all tournaments T, T_1, \dots, T_k , and \tilde{T} such that $T = \Pi(\tilde{T}, T_1, \dots, T_k)$, $S(T) = \bigcup_{i \in S(\tilde{T})} S(T_i)$.

Finally, a tournament solution is irregular if it is capable of excluding alternatives in regular tournaments.

Definition 11. A tournament solution S satisfies *irregularity* (IRR) if there exists a regular tournament $T = (A, \succ)$ such that $S(T) \neq A$.

3 Qualified Subsets

In this section, we will define a class of tournament solutions that is based on identifying significant subtournaments of the original tournament, such as maximal subtournaments that admit a maximal alternative.

3.1 Concepts of Qualified Subsets

A *concept of qualified subsets* is a tournament function that, for a given tournament $T = (A, \succ)$, returns subsets of A that satisfy certain properties. Each such set of sets

will be referred to as a *family of qualified subsets*. Two natural examples of concepts of qualified subsets are \mathcal{M} , which yields all subsets that admit a maximum, and \mathcal{M}^* , which yields all non-empty transitive subsets. Formally,

$$\begin{aligned}\mathcal{M}((A, \succ)) &= \{B \subseteq A \mid \max_{\prec}(B) \neq \emptyset\}, \\ \mathcal{M}^*((A, \succ)) &= \{B \subseteq A \mid \max_{\prec}(C) \neq \emptyset \text{ for all non-empty } C \subseteq B\}.\end{aligned}$$

For given $k \in \mathbb{N}$, we will also consider bounded variants of \mathcal{M} and \mathcal{M}^* , viz. the tournament functions \mathcal{M}_k and \mathcal{M}_k^* such that $\mathcal{M}_k(T) = \{B \in \mathcal{M}(T) \mid |B| \leq k\}$ and $\mathcal{M}_k^*(T) = \{B \in \mathcal{M}^*(T) \mid |B| \leq k\}$ for all tournaments T . Furthermore, the short notation $[B, a]$ will be used to denote the set $B \uplus \{a\}$ with $\max_{\prec}(B \uplus \{a\}) = \{a\}$.

\mathcal{M} , \mathcal{M}^* , \mathcal{M}_k , and \mathcal{M}_k^* are all examples of concepts of qualified subsets, which are defined as follows.

Definition 12. Let $\mathcal{Q} : \mathcal{T}(X) \rightarrow \mathcal{F}(\mathcal{F}(X))$ be a tournament function such that $\mathcal{M}_1(T) \subseteq \mathcal{Q}(T) \subseteq \mathcal{M}(T)$. \mathcal{Q} is a *concept of qualified subsets* if it meets the following three conditions for any tournament $T = (A, \succ)$.

(Closure) $\mathcal{Q}(T)$ is downward closed with respect to \mathcal{M} : Let $Q \in \mathcal{Q}(T)$. Then, $Q' \in \mathcal{Q}(T)$ if $Q' \subseteq Q$ and $Q' \in \mathcal{M}(T)$.

(Independence) Qualified sets are independent of outside alternatives: Let $A' \in \mathcal{F}(X)$ and $Q \subseteq A \cap A'$. Then, $Q \in \mathcal{Q}(A)$ if and only if $Q \in \mathcal{Q}(A')$.

(Fusion) Qualified sets may be merged under certain conditions: Let $[Q_1, a], [Q_2, a] \in \mathcal{Q}(T)$. Then $[Q_1 \cup Q_2, a] \in \mathcal{Q}(T)$ if $Q_1 \setminus Q_2 \succ Q_2$ and there is a tournament $T' \in \mathcal{T}(X)$ and $Q \in \mathcal{Q}(T')$ such that $|Q_1 \cup Q_2 \cup \{a\}| \leq |Q|$.⁴

Note that, in particular, all singletons are qualified. Whether a set is qualified only depends on its internal structure (due to independence and the isomorphism condition of Definition 4).

For any concept of qualified subsets \mathcal{Q} and any given $k \in \mathbb{N}$, $\mathcal{Q}_k : \mathcal{T}(X) \rightarrow \mathcal{F}(\mathcal{F}(X))$ is a tournament function such that $\mathcal{Q}_k(T) = \{B \in \mathcal{Q}(T) \mid |B| \leq k\}$. It is easily verified that \mathcal{Q}_k is a concept of qualified subsets. Furthermore, \mathcal{M} and \mathcal{M}^* (and thus also \mathcal{M}_k and \mathcal{M}_k^*) are concepts of qualified subsets. Since only tournaments of order 4 or more may be intransitive and admit a maximal element at the same time, $\mathcal{M}_k = \mathcal{M}_k^*$ for $k \in \{1, 2, 3\}$.

3.2 Maximal Elements of Maximal Qualified Subsets

For any concept of qualified subsets, we can now define a tournament solution that yields the maximal elements of all inclusion-maximal qualified subsets, i.e., all qualified subsets that are not contained in another qualified subset.

⁴The latter condition is only required to enable bounded qualified subsets.

Definition 13. Let \mathcal{Q} be a concept of qualified subsets. Then, the tournament solution $S_{\mathcal{Q}}$ is defined as

$$S_{\mathcal{Q}}(T) = \{\max_{\prec}(B) \mid B \in \max_{\subseteq}(\mathcal{Q}(T))\}.$$

Since any family of qualified subsets contains all singletons, $S_{\mathcal{Q}}(T)$ is guaranteed to be non-empty and contains the Condorcet winner whenever one exists. As a consequence, $S_{\mathcal{Q}}$ is well-defined as a tournament solution. In several proofs, we will make use of the following fact, which follows from closure: Whenever $a \notin S_{\mathcal{Q}}(T)$, there is some $b \in S_{\mathcal{Q}}(T)$ for every qualified subset $[Q, a]$ such that $b \succ [Q, a]$ and $[Q \cup \{a\}, b] \in \mathcal{Q}(T)$. The following tournament solutions can be restated via appropriate concepts of qualified subsets.

Condorcet non-losers. $S_{\mathcal{M}_2}$ is arguably the largest non-trivial tournament solution. It chooses every alternative that dominates at least one other alternative. We will sometimes refer to this concept as *Condorcet non-losers (CNL)* as it selects everything except the minimum (or *Condorcet loser*) whenever such a minimum exists and there is more than one alternative.

Uncovered set. $S_{\mathcal{M}}(T)$ returns the *uncovered set* $UC(T)$ of a tournament T , i.e., the set consisting of the maximal elements of inclusion-maximal subsets that admit a maximum. This concept is usually defined in terms of a subrelation of the dominance relation called the *covering relation* (Fishburn, 1977; Miller, 1980).

Banks set. $S_{\mathcal{M}^*}(T)$ yields the *Banks set* $BA(T)$ of a tournament T (Banks, 1985). $\mathcal{M}^*(T)$ contains subsets that not only admit a maximum, but can be completely ordered from maximum to minimum so that all of their non-empty subsets admit a maximum. $S_{\mathcal{M}^*}(T)$ thus returns the maximal elements of inclusion-maximal *transitive* subsets.

Every tournament solution defined via qualified subsets satisfies the weak superset property and monotonicity.

Proposition 1. *Let \mathcal{Q} be a concept of qualified subsets. Then, $S_{\mathcal{Q}}$ satisfies WSP and MON.*

Proof. Let $T = (A, \succ)$ be a tournament, $a \notin S_{\mathcal{Q}}(A)$, and $A' \subseteq A$ such that $S_{\mathcal{Q}}(T) \cup \{a\} \subseteq A'$. For WSP, we need to show that $a \notin S_{\mathcal{Q}}(A')$. Let $[Q, a] \in \mathcal{Q}(A')$. Due to independence, $[Q, a] \in \mathcal{Q}(A)$. Since $a \notin S_{\mathcal{Q}}(A)$, there has to be some $b \in S_{\mathcal{Q}}(A)$ such that $[Q \cup \{a\}, b] \in \mathcal{Q}(A)$. Again, independence implies that $[Q \cup \{a\}, b] \in \mathcal{Q}(A')$. Hence, $a \notin S_{\mathcal{Q}}(A')$.

For MON, observe that $a \in S_{\mathcal{Q}}$ implies that there exists $[Q, a] \in \max_{\subseteq}(\mathcal{Q}(T))$. Define $T' = (A, \succ')$ by letting $T'|_{A \setminus \{a\}} = T|_{A \setminus \{a\}}$ and $a \succ' b$ for some $b \in A$ with $b \succ a$. Clearly, $[Q, a]$ is contained in $\mathcal{Q}(T')$ due to independence and the fact that $b \notin Q$. Now, assume for contradiction that there is some $c \in A$ such that $[Q \cup \{a\}, c] \in \mathcal{Q}(T')$. Since $a \succ' b$, $c \neq b$. Independence then implies that $[Q \cup \{a\}, c] \in \mathcal{Q}(T)$, a contradiction. \square

Proposition 1 implies several known statements such as that *CNL*, *UC*, and *BA* satisfy *MON* and *WSP*. All three concepts are known to fail *IDE* (and thus *SSP*). *CNL* trivially satisfies *IUA* whereas this is not the case for *UC* and *BA* (see Laslier, 1997). We also obtain some straightforward inclusion relationships, which define an infinite hierarchy of tournament solutions ranging from *CNL* to *BA*.

Proposition 2. $S_{\mathcal{M}^*} \subseteq S_{\mathcal{M}}$, $S_{\mathcal{M}_k^*} \subseteq S_{\mathcal{M}_k}$, $S_{\mathcal{Q}_{k+1}} \subseteq S_{\mathcal{Q}_k}$, and $S_{\mathcal{Q}_k} \subseteq S_{\mathcal{Q}}$ for any concept of qualified subsets \mathcal{Q} and $k \in \mathbb{N}$.

Proof. All inclusion relationships follow from the following observation. Let \mathcal{Q} and \mathcal{Q}' be concepts of qualified subsets such that for every $[Q, a] \in \max_{\subseteq}(\mathcal{Q})$, there is $[Q', a] \in \max_{\subseteq}(\mathcal{Q}')$. Then, $S_{\mathcal{Q}} \subseteq S_{\mathcal{Q}'}$. \square

It turns out that the Banks set is the finest tournament solution definable via qualified subsets. In order to show this, we introduce a variant of *WSP* that we call *dominion contraction*. Dominion contraction prescribes that a choice set may not grow when an alternative and its entire dominion are removed from the tournament.

Definition 14. A tournament solution S satisfies *dominion contraction* (*DCON*) if $S(A \setminus D_{\succeq}(a)) \subseteq S(A)$ for all tournaments $T = (A, \succ)$ and $a \in A$.

Since $A \setminus D_{\succeq}(a) = \overline{D}(a)$, an alternative definition of *DCON* requires that $S(\overline{D}(a)) \subseteq S(A)$ for all $a \in A$, i.e., the choice set of every dominator set is contained in the original choice set. It follows from the non-emptiness of choice sets, that at least one alternative from any dominator set is contained in $S(A)$.

Lemma 1. *Let \mathcal{Q} be a concept of qualified subsets. Then, $S_{\mathcal{Q}}$ satisfies *DCON*.*

Proof. Let (A, \succ) be a tournament and $a \in A$ an alternative. Further let $B = A \setminus D_{\succeq}(a)$. We show that $b \in S_{\mathcal{Q}}(B)$ implies that $b \in S_{\mathcal{Q}}(A)$. Let $[Q, b]$ be a maximal qualified subset in B , i.e., $[Q, b] \in \max_{\subseteq}(\mathcal{Q}(B))$. If $[Q, b] \in \max_{\subseteq}(\mathcal{Q}(A))$, we are done. Otherwise, there has to be some $c \in A$ such that $c \succ [Q, b]$ and $[Q \cup \{b\}, c] \in \mathcal{Q}(A)$. Furthermore, $[Q, b] \succ a$ and $a \succ c$ because otherwise $[Q \cup \{b\}, c]$ would be qualified in B as well. We can now merge the qualified subsets $[Q, b]$ and $[\{a\}, b]$ according to the fusion condition. We claim that $[Q \cup \{a\}, b] \in \max_{\subseteq}(\mathcal{Q}(A))$. Assume for contradiction that there is some $d \in A$ such that $d \succ [Q \cup \{a\}, b]$ and $[Q \cup \{a, b\}, d] \in \mathcal{Q}(A)$. Since $d \notin D_{\succeq}(a)$, Independence implies that $[Q \cup \{a, b\}, d] \in \mathcal{Q}(B)$. This is a contradiction because $[Q, b]$ was assumed to be a *maximal* qualified subset of B . \square

Theorem 1. *The Banks set is the finest tournament solution satisfying *DCON*.*

Proof. Let S be a tournament solution that satisfies *DCON* and $T = (A, \succ)$ a tournament. We first show that $BA(T) \subseteq S(T)$. For any $a \in BA(T)$, there has to be maximal transitive set $[Q, a] = \{q_1, q_2, \dots, q_n\} \subseteq A$ such that $q_i \succ q_j$ for all $1 \leq j < i \leq n$. We show that $B = (((A \setminus D_{\succeq}(q_1)) \setminus D_{\succeq}(q_2)) \setminus \dots) \setminus D_{\succeq}(q_n) = \{a\}$. Since $a \succ Q$, $a \in B$. Assume for contradiction that $b \in B$ with $b \neq a$. Then $b \succ Q$ and either $[Q \cup \{b\}, a]$ or $[Q \cup \{a\}, b]$ is a transitive set, which contradicts the maximality of $[Q, a]$. Since $S(B) = S(\{a\}) = \{a\}$, the repeated application of *DCON* implies that $a \in S(A)$. The statement now follows from Lemma 1. \square

Since all tournament solutions S_Q for some concept of qualified subsets Q satisfy DCON, $BA \subseteq S_Q$.

4 Stable Sets

In this section, we propose a general method for refining tournament solutions by formalizing the stability of sets of alternatives with respect to some underlying solution concept. This method is based on the notion of *stable sets* introduced by von Neumann and Morgenstern (1944) and generalizes the concept of a minimal covering set by Dutta (1988).

4.1 Stability and Directedness

The intuition underlying stable sets is that any choice set should comply with internal and external stability in some well-defined way. First, there should be no reason to restrict the selection by excluding some alternative from it and, secondly, there should be an argument against each proposal to include an outside alternative into the selection.⁵ In our context, external stability with respect to some tournament solution S is defined as follows.

Definition 15. Let S be a tournament solution and $T = (A, \succ)$ a tournament. Then, $B \subseteq A$ is *externally stable* in T with respect to tournament solution S (or *S -stable*) if $a \notin S(B \cup \{a\})$ for all $a \in A \setminus B$. The set of S -stable sets for a given tournament $T = (A, \succ)$ will be denoted by $\mathcal{S}_S(T) = \{B \subseteq A \mid B \text{ is } S\text{-stable in } T\}$.

Externally stable sets are guaranteed to exist since the set of all alternatives A is trivially S -stable in (A, \succ) for any S . We say that a set $B \subseteq A$ is *internally stable* with respect to S if $S(B) = B$. We will focus on external stability for now because we will see later that certain conditions imply the existence of a unique *minimal* externally stable set, which also satisfies internal stability. We define $\hat{S}(T)$ to be the tournament solution that returns the union of all *inclusion-minimal* S -stable sets in T , i.e., the union of all S -stable sets that do not contain an S -stable set as a proper subset.

Definition 16. Let S be a tournament solution. Then, the tournament solution \hat{S} is defined as

$$\hat{S}(T) = \bigcup_{\subseteq} \min(\mathcal{S}_S(T)).$$

It is easily verified that \hat{S} is well-defined as a tournament solution as there are no S -stable sets that do not contain the Condorcet winner whenever one exists. We will only be concerned with tournament solutions S that (presumably) admit a *unique* minimal S -stable set in any tournament. It turns out it is precisely this property that is most

⁵A large number of solution concepts in the social sciences spring from similar notions of internal and/or external stability (see, e.g., von Neumann and Morgenstern, 1944; Nash, 1951; Shapley, 1964; Schwartz, 1986; Dutta, 1988; Basu and Weibull, 1991; Duggan and Le Breton, 1996). Wilson (1970) refers to stability as the *solution property*.

difficult to prove for all but the simplest tournament solutions. A tournament T contains a *unique* minimal S -stable set if and only if $\mathcal{S}_S(T)$ is a *directed* set, i.e., for any two sets $B, C \in \mathcal{S}_S(T)$ there is a set $D \in \mathcal{S}_S(T)$ contained in both B and C . We say that \mathcal{S}_S is directed when $\mathcal{S}_S(T)$ is a directed set for all tournaments T . Throughout this paper, directedness of a set of sets \mathcal{S} is shown by proving the stronger property of closure under intersection, i.e., $B \cap C \in \mathcal{S}$ for all $B, C \in \mathcal{S}$. A set of sets \mathcal{S} *pairwise intersects* if $B \cap C \neq \emptyset$ for all $B, C \in \mathcal{S}$. We will prove that, for any concept of qualified subsets \mathcal{Q} , $\mathcal{S}_{\mathcal{S}_{\mathcal{Q}}}$ is closed under intersection if and only if $\mathcal{S}_{\mathcal{Q}}$ pairwise intersects. In order to improve readability in the following, we will use the short notation $\mathcal{S}_{\mathcal{Q}}$ for $\mathcal{S}_{\mathcal{S}_{\mathcal{Q}}}$.

Theorem 2. *Let \mathcal{Q} be a concept of qualified subsets. Then, $\mathcal{S}_{\mathcal{Q}}$ is closed under intersection if and only if $\mathcal{S}_{\mathcal{Q}}$ pairwise intersects.*

Proof. The direction from left to right is straightforward since the empty set is not stable. The opposite direction is shown by contraposition, i.e., we prove that $\mathcal{S}_{\mathcal{Q}}$ does not pairwise intersect if $\mathcal{S}_{\mathcal{Q}}$ is not closed under intersection. Let $T = (A, \succ)$ be a tournament and $B_1, B_2 \in \mathcal{S}_{\mathcal{Q}}(T)$ be two sets such that $C = B_1 \cap B_2 \notin \mathcal{S}_{\mathcal{Q}}(T)$. Since C is not $\mathcal{S}_{\mathcal{Q}}$ -stable, there has to be $a \in A \setminus C$ such that $a \in \mathcal{S}_{\mathcal{Q}}(C \cup \{a\})$. In other words, there has to be a set $Q \subseteq C$ such that $[Q, a] \in \max_{\subseteq}(\mathcal{Q}(C \cup \{a\}))$. Define

$$B'_1 = \{b \in B_1 \mid b \succ Q\} \text{ and } B'_2 = \{b \in B_2 \mid b \succ Q\}.$$

Clearly, $(B'_1 \setminus B'_2) \cap C = \emptyset$ and $(B'_2 \setminus B'_1) \cap C = \emptyset$. Assume without loss of generality that $a \notin B_1$. It follows from the stability of B_1 , that B_1 has to contain an alternative b_1 such that $b_1 \succ [Q, a]$. Hence, B'_1 is not empty. Next, we show that $B'_1 \cap B'_2 = \emptyset$. Assume for contradiction that $b \in B'_1 \cap B'_2$. If $b \succ a$, independence implies that $[Q \cup \{a\}, b] \in \mathcal{Q}(C \cup \{a\})$, which contradicts the fact that $[Q, a]$ is a maximal qualified subset in $C \cup \{a\}$. If, on the other hand, $a \succ b$, the set $[Q \cup \{b\}, a]$ is isomorphic to $[Q \cup \{a\}, b_1]$, which is a qualified subset of $B_1 \cup \{a\}$. Thus, $[Q \cup \{b\}, a] \in \mathcal{Q}(C \cup \{a\})$, again contradicting the maximality of $[Q, a]$. Independence, the isomorphism of $[Q, a]$ and $[Q, b_1]$, and the stability of B_2 further require that there has to be an alternative $b_2 \in B_2$ such that $b_2 \succ [Q, b_1]$. Hence, B'_1 and B'_2 are disjoint and non-empty.

Let $a' \in B'_2$ and R be a maximal subset of $B'_1 \cup Q$ such that $[R, a'] \in \mathcal{Q}(B'_1 \cup Q \cup \{a'\})$. We claim that Q has to be contained in R . Assume for contradiction that there exists $b \in Q \setminus R$. Clearly, $[Q, a]$ and $[Q, a']$ are isomorphic. It therefore follows from independence that $[Q, a'] \in \mathcal{Q}(B'_1 \cup Q \cup \{a'\})$ and from closure that $[(Q \cap R) \cup \{b\}, a'] \in \mathcal{Q}(B'_1 \cup Q \cup \{a'\})$. Due to the stability of B_1 , $[R, a']$ is not a maximal qualified subset in $B_1 \cup \{a'\}$, i.e., there exists a qualified subset that contains more elements. We may thus merge the qualified subsets $[R, a']$ and $[(Q \cap R) \cup \{b\}, a']$ according to the fusion condition because $R \setminus Q \succ Q$ and consequently $R \setminus Q \succ (Q \cap R) \cup \{b\}$. We then have that $[R \cup \{b\}, a'] \in \mathcal{Q}(B'_1 \cup Q \cup \{a'\})$, which yields a contradiction because R was assumed to be a maximal set such that $[R, a'] \in \mathcal{Q}(B'_1 \cup Q \cup \{a'\})$. Hence, $Q \subseteq R$. Due to the stability of B_1 in T , there has to be a $c \in B_1$ such that $c \succ [R, a']$. Since B'_1 contains all alternatives in B_1 that dominate $Q \subseteq R$, it also contains c . Independence then implies that $[R, a'] \notin \max_{\subseteq}(\mathcal{Q}_k(B'_1 \cup Q \cup \{a'\}))$.

Thus, $B'_1 \cup Q$ is stable in $B'_1 \cup B'_2 \cup Q$. Since Q is contained in every maximal set $R \subseteq B'_1 \cup Q$ such that $[R, a'] \in \mathcal{Q}(B'_1 \cup Q \cup \{a'\})$ for some $a' \in B'_2$, B'_1 (and by an analogous argument B'_2) remains stable when removing Q . This completes the proof because B'_1 and B'_2 are two disjoint $S_{\mathcal{Q}}$ -stable sets in $B'_1 \cup B'_2$. \square

Dutta has shown by induction on the tournament order that tournaments admit no disjoint $S_{\mathcal{M}}$ -stable sets (so-called *covering sets*).

Theorem 3 (Dutta, 1988). $S_{\mathcal{M}}$ pairwise intersects.

Dutta (1988) also showed that covering sets are closed under intersection, which now also follows from Theorem 2.⁶

Naturally, finer solution concepts also yield smaller minimal stable sets (if their uniqueness is guaranteed).

Proposition 3. Let S and S' be two tournament solutions such that $S_{S'}$ is directed and $S' \subseteq S$. Then, $\widehat{S}' \subseteq \widehat{S}$ and S_S pairwise intersects.

Proof. The statements follow from the simple fact that every S -stable set is also S' -stable. Let $B \subseteq A$ be a minimal S -stable set in tournament (A, \succ) . Then, $a \notin S(B \cup \{a\})$ for any $a \in A \setminus B$ and, due to the inclusion relationship, $a \notin S'(B \cup \{a\}) \subseteq S(B \cup \{a\})$. As a consequence, B is S' -stable and has to contain the unique minimal S' -stable set since $S_{S'}$ is directed. S_S pairwise intersects because two disjoint S -stable sets would also be S' -stable, which contradicts the directedness of $S_{S'}$. \square

As a corollary of the previous statements, the set of $S_{\mathcal{M}_k}$ -stable sets for any k is closed under intersection.

Theorem 4. $S_{\mathcal{M}_k}$ is closed under intersection for all $k \in \mathbb{N}$.

Proof. Let $k \in \mathbb{N}$. We know from Proposition 2 that $S_{\mathcal{M}} \subseteq S_{\mathcal{M}_k}$ and from Theorem 3 and Theorem 2 that $S_{\mathcal{M}}$ is directed. Proposition 3 implies that $S_{\mathcal{M}_k}$ pairwise intersects. The statement then straightforwardly follows from Theorem 2. \square

Interestingly, $S_{\mathcal{M}_2}$, the set of all dominating sets, is not only closed under intersection, but in fact totally ordered with respect to set inclusion.

We conjecture that the set of all $S_{\mathcal{M}^*}$ -stable sets also pairwise intersects and thus admits a unique minimal element. However, the combinatorial structure of transitive subtournaments within tournaments is surprisingly rich and it seems that a proof of the conjecture would be significantly more difficult than Dutta's. So far, it is not even known whether $S_{\mathcal{M}_4^*}$ -stable sets pairwise intersect. As will be shown in Section 5, the conjecture that $S_{\mathcal{M}^*}$ -stable sets pairwise intersect is a weakened version of a conjecture by Schwartz (1990).

⁶As Dutta's definition requires a stable set to be internally and externally stable, he actually proves that the intersection of any pair of coverings sets *contains* a covering set. A simpler proof, which shows that *externally* $S_{\mathcal{M}}$ -stable sets are closed under intersection, is given by Laslier (1997).

Conjecture 1. $\mathcal{S}_{\mathcal{M}^*}$ is closed under intersection.

Using Theorem 2, the conjecture entails that $\mathcal{S}_{\mathcal{M}_k^*}$ for any $k \in \mathbb{N}$ is also closed under intersection. Since tournaments with less than four alternatives may not contain a maximal element and a cycle at the same time, this trivially holds for $k \leq 3$.

Two well-known examples of minimal stable sets are the top cycle of a tournament, which is the minimal stable set with respect to $\mathcal{S}_{\mathcal{M}_2}$, and the minimal covering set, which is the minimal stable set with respect to $\mathcal{S}_{\mathcal{M}}$.

Minimal dominating set. The *minimal dominating set* (or top cycle) of a tournament $T = (A, \succ)$ is given by $TC(T) = \widehat{S}_{\mathcal{M}_2}(T) = \widehat{CNL}$, i.e., it is the smallest set B such that $B \succ A \setminus B$ (Good, 1971; Smith, 1973).

Minimal covering set. The *minimal covering set* of a tournament T is given by $MC(T) = \widehat{S}_{\mathcal{M}}(T) = \widehat{UC}$, i.e., it is the smallest set B such that for all $b \in A \setminus B$, there exists $a \in B$ so that every alternative in B that is dominated by b is also dominated by a (Dutta, 1988).

The proposed methodology also suggests the definition of a new tournament solution that has not been considered before in the literature.

Minimal extending set. The *minimal extending set* of a tournament T is given by $ME(T) = \widehat{S}_{\mathcal{M}^*}(T) = \widehat{BA}$, i.e., it is the smallest set B such that no $a \in A \setminus B$ is the maximal element of a maximal transitive subset in $B \cup \{a\}$.

The minimal extending set will be further analyzed in Section 4.3.

4.2 Properties of Minimal Stable Sets

If \mathcal{S}_S is directed—and we will only be concerned with tournament solutions S for which this is (presumably) the case— \widehat{S} satisfies a number of desirable properties.

Proposition 4. *Let S be a tournament solution such that \mathcal{S}_S is directed. Then, \widehat{S} satisfies WSP and IUA.*

Proof. Clearly, any minimal S -stable set B remains S -stable when losing alternatives are removed or when edges between losing alternatives are modified. In the latter case, B also remains minimal. In the former case, the minimal S -stable set is contained in B . \square

It can be shown that the stable sets of a stable set are stable when the underlying tournament solution is defined via a concept of qualified subsets \mathcal{Q} . This lemma will prove very useful when analyzing $\widehat{S}_{\mathcal{Q}}$.

Lemma 2. *Let $T = (A, \succ)$ be a tournament and \mathcal{Q} a concept of qualified subsets. Then, $\mathcal{S}_{\mathcal{Q}}(B) \subseteq \mathcal{S}_{\mathcal{Q}}(A)$ for all $B \in \mathcal{S}_{\mathcal{Q}}(A)$.*

Proof. We prove the statement by showing that the following implication holds for all $B \subseteq A$, $C \in \mathcal{S}_{\mathcal{Q}}(B)$, and $a \in A$:

$$a \notin S_{\mathcal{Q}}(B \cup \{a\}) \implies a \notin S_{\mathcal{Q}}(C \cup \{a\}).$$

To see this, let $a \notin S_{\mathcal{Q}}(B \cup \{a\})$ and assume for contradiction that there exist $[Q, a] \in \max_{\subseteq} \mathcal{Q}(C \cup \{a\})$. Then there has to be $b \in B$ such that $b \succ [Q, a]$ and $[Q \cup \{a\}, b] \in \mathcal{Q}(B \cup \{a\})$ because $[Q, a] \notin \max_{\subseteq} \mathcal{Q}(B \cup \{a\})$. Now, if $b \in C$, closure and independence imply that $[Q \cup \{a\}, b] \in \mathcal{Q}(C \cup \{a\})$, contradicting the maximality of $[Q, a]$. If, on the other hand, $b \in B \setminus C$, then there has to be $c \in C$ such that $c \succ [Q, b]$ and $[Q \cup \{b\}, c] \in \mathcal{Q}(C \cup \{b\})$. No matter whether $c \succ a$ or $a \succ c$, $Q \cup \{a, c\}$ is isomorphic to $[Q \cup \{b\}, c]$ and thus also a qualified subset, which again contradicts the assumption that $[Q, a]$ was maximal. \square

Theorem 5. *Let \mathcal{Q} be a concept of qualified subsets such that $\mathcal{S}_{\mathcal{Q}}$ is directed. Then,*

- (i) $\widehat{\mathcal{S}}_{\mathcal{Q}} \subseteq S_{\mathcal{Q}}^{\infty}$,
- (ii) $S_{\mathcal{Q}}(\widehat{\mathcal{S}}_{\mathcal{Q}}(T) \cup \{a\}) = \widehat{\mathcal{S}}_{\mathcal{Q}}(T)$ for all tournaments $T = (A, \succ)$ and $a \in A$ (in particular, $\widehat{\mathcal{S}}_{\mathcal{Q}}(T)$ is internally stable),
- (iii) $\widehat{\mathcal{S}}_{\mathcal{Q}}$ satisfies SSP, and
- (iv) $\widehat{\widehat{\mathcal{S}}}_{\mathcal{Q}} = \widehat{\mathcal{S}}_{\mathcal{Q}}$.

Proof. Let $T = (A, \succ)$ be a tournament. The first statement of the theorem is shown by proving by induction on k that $S_{\mathcal{Q}}^k(T)$ is an $S_{\mathcal{Q}}$ -stable set. For the basis, let $B = S_{\mathcal{Q}}(T)$. Then, $S_{\mathcal{Q}}(B \cup \{a\}) \subseteq B$ for any $a \in A \setminus B$ due to WSP of $S_{\mathcal{Q}}$ (Proposition 1) and thus B is $S_{\mathcal{Q}}$ -stable. Now, assume that $B = S_{\mathcal{Q}}^k(T)$ is $S_{\mathcal{Q}}$ -stable and let $C = S_{\mathcal{Q}}(B)$. Again, WSP implies that $a \notin S_{\mathcal{Q}}(C \cup \{a\})$ for any $a \in B \setminus C$, i.e., $C \in \mathcal{S}_{\mathcal{Q}}(B)$. We can thus directly apply Lemma 2 to obtain that $C = S_{\mathcal{Q}}^{k+1}(T) \in \mathcal{S}_{\mathcal{Q}}(T)$. As the minimal $S_{\mathcal{Q}}$ -stable set is contained in every $S_{\mathcal{Q}}$ -stable set, the statement follows.

Regarding internal stability, assume for contradiction that $S_{\mathcal{Q}}(\widehat{\mathcal{S}}_{\mathcal{Q}}(T)) \subset \widehat{\mathcal{S}}_{\mathcal{Q}}(T)$. However, Lemma 2 implies that $S_{\mathcal{Q}}(\widehat{\mathcal{S}}_{\mathcal{Q}}(T))$ is $S_{\mathcal{Q}}$ -stable, contradicting the minimality of $\widehat{\mathcal{S}}_{\mathcal{Q}}(T)$. The remainder of the second statement follows straightforwardly from internal stability. If $S_{\mathcal{Q}}(\widehat{\mathcal{S}}_{\mathcal{Q}}(T) \cup \{a\}) = C \subset \widehat{\mathcal{S}}_{\mathcal{Q}}(T)$ for some $a \in A \setminus \widehat{\mathcal{S}}_{\mathcal{Q}}(T)$, WSP implies that $S_{\mathcal{Q}}(\widehat{\mathcal{S}}_{\mathcal{Q}}(T)) \subseteq C$, contradicting internal stability.

Regarding SSP, let $B = \widehat{\mathcal{S}}_{\mathcal{Q}}(T)$ and assume for contradiction that $C = \widehat{\mathcal{S}}_{\mathcal{Q}}(A') \subset B$ for some A' with $B \subseteq A' \subset A$. Clearly, C is $S_{\mathcal{Q}}$ -stable not only in A' but also in B , which implies that $C \in \mathcal{S}_{\mathcal{Q}}(B)$. According to Lemma 2, C is also contained in $\mathcal{S}_{\mathcal{Q}}(A)$, contradicting the minimality of $\widehat{\mathcal{S}}_{\mathcal{Q}}(T)$.

Finally, for $\widehat{\widehat{\mathcal{S}}}_{\mathcal{Q}}(T) = \widehat{\mathcal{S}}_{\mathcal{Q}}(T)$, we show that every $S_{\mathcal{Q}}$ -stable set is $\widehat{\mathcal{S}}_{\mathcal{Q}}$ -stable and that every *minimal* $\widehat{\mathcal{S}}_{\mathcal{Q}}$ -stable set is $S_{\mathcal{Q}}$ -stable set. The former follows from $\widehat{\mathcal{S}}_{\mathcal{Q}}(T) \subseteq S_{\mathcal{Q}}(T)$, which was shown at the beginning of this proof. For the latter statement, let $B \in \mathcal{S}_{\widehat{\mathcal{S}}_{\mathcal{Q}}}(T)$. We first show that $\widehat{\mathcal{S}}_{\mathcal{Q}}(B \cup \{a\}) = B$ for all $a \in A \setminus B$. Assume for contradiction that

$\widehat{S}_Q(B \cup \{a\}) = C \subset B$ for some $a' \in A \setminus B$. Since \widehat{S}_Q satisfies SSP, $\widehat{S}_Q(B \cup \{a\}) = C$ for all $a \in A \setminus B$. As a consequence, C is \widehat{S}_Q -stable in $B \cup \{a\}$ for all $a \in A \setminus B$ and, due to the definition of stability, also in A . This contradicts the assumption that B was the *minimal* \widehat{S} -stable set. Hence, $\widehat{S}_Q(B \cup \{a\}) = B$ for all $a \in A \setminus B$. By definition of \widehat{S}_Q , this implies that $a \notin S_Q(B \cup \{a\})$ and thus that B is S_Q -stable. \square

The second statement of Theorem 5 yields the following fixed-point characterization of \widehat{S}_Q , unifying internal and external stability.

Corollary 1. *Let Q be a concept of qualified subsets such that S_Q is directed and $T = (A, \succ)$ a tournament. Then $\widehat{S}_Q(T)$ is the unique inclusion-minimal set $B \subseteq A$ such that*

$$B = \{a \in A \mid a \in S_Q(B \cup \{a\})\}.$$

There may very well be more than one internally and externally S_Q -stable set. For example, Theorem 5 implies that $S_Q^\infty(T)$ is internally and externally S_Q -stable.

We have already seen that \widehat{S}_Q satisfies some of the properties defined in Section 2.5. It further turns out that \widehat{S} inherits monotonicity and composition-consistency from S .

Proposition 5. *Let S be a tournament solution such that S_S is directed and S satisfies MON. Then, \widehat{S} satisfies MON as well.*

Proof. First observe that monotonicity implies that weakening a losing alternative cannot make this alternative a winner. Formally, $a \notin S(T)$ implies $a \notin S(T')$ for any two tournaments $T = (A, \succ)$ and $T' = (A, \succ')$ with $\succ|_{A \setminus \{a\}} = \succ'|_{A \setminus \{a\}}$ and $D_{\succ'}(a) \subseteq D_\succ(a)$.

Now, let $T = (A, \succ)$ be a tournament with $a, b \in A$, $a \in \widehat{S}(T)$, and $b \succ a$, and let the relation \succ' be identical to \succ except that $a \succ' b$. Denote $T' = (A, \succ')$ and assume for contradiction that $a \notin \widehat{S}(T')$. Then, there has to be some S -stable set $B \subseteq A \setminus \{a\}$ in T' . If $b \notin B$, then B would also be an S -stable set in T , contradicting $a \in \widehat{S}(T)$. If, on the other hand, $b \in B$, B would also be S -stable in T since weakening a cannot make it a winner, which implies $a \notin S((B \cup \{a\}, \succ))$. \square

Proposition 6. *Let S be a tournament solution that satisfies COM. Then, \widehat{S} satisfies COM as well.*

Proof. Let S be a composition-consistent tournament solution and $T = (A, \succ) = \Pi(\tilde{T}, T_1, \dots, T_k)$ a product tournament with $T = (\{1, \dots, k\}, \tilde{\succ})$, $T_1 = (B_1, \succ_1)$, \dots , $T_k = (B_k, \succ_k)$. For a subset C of A , let $C_i = C \cap B_i$ for all $i \in \{1, \dots, k\}$ and $\tilde{C} = \bigcup_{i: C_i \neq \emptyset} \{i\}$. We will prove that $C \subseteq A$ is S -stable if and only if

- (i) \tilde{C} is S -stable in \tilde{T} , and
- (ii) C_i is S -stable in T_i for any $i \in \{1, \dots, k\}$.

Consider an arbitrary alternative $a \in A \setminus C$. For C to be S -stable, a should not be contained in $S(C \cup \{a\})$. Since S is composition-consistent, a may be excluded for two reasons. First, a may be contained in an unchosen component, i.e., $a \in B_i$ such that

$i \notin S(\tilde{C} \cup \{i\})$. Secondly, a may not be selected despite being in a chosen component, i.e., $a \in B_i$ such that $i \in S(\tilde{C} \cup \{i\})$ and $a \notin S(C_i \cup \{a\})$. This directly establishes the claim above and consequently that \hat{S} is composition-consistent. \square

Propositions 4, 5, 6, Theorem 5, and Theorem 4 allow us to deduce several known statements about TC and MC , in particular that both concepts satisfy all basic properties and that MC is a refinement of UC^∞ and satisfies COM.

We conclude this section by generalizing the axiomatization of the minimal covering set (Dutta, 1988) to abstract minimal stable sets. One of the cornerstones of the axiomatization is S -exclusivity, which prescribes under which circumstances a single element may be dismissed from the choice set.⁷

Definition 17. A tournament solution S' satisfies S -exclusivity if, for any tournament $T = (A, \succ)$, $S'(T) = A \setminus \{a\}$ implies that $a \notin S(A)$.

If S always admits a *unique* minimal S -stable set and \hat{S} satisfies SSP, which is always the case if S is defined via qualified subsets, then \hat{S} can be characterized by SSP, S -exclusivity, and inclusion-minimality.

Proposition 7. Let S be a tournament solution such that S_S is directed and \hat{S} satisfies SSP. Then, \hat{S} is the finest tournament solution satisfying SSP and S -exclusivity.

Proof. Let S be a tournament solution as desired and S' a tournament solution that satisfies SSP and S -exclusivity. We first prove that $\hat{S} \subseteq S'$ by showing that $S'(T)$ is S -stable for any tournament $T = (A, \succ)$. Let $B = S'(A)$ and $a \in A \setminus B$. It follows from SSP that $S'(B \cup \{a\}) = B$ and from S -exclusivity that $a \notin S(B \cup \{a\})$, which implies that B is S -stable. Since \hat{S} is the unique inclusion-minimal S -stable set, it has to be contained in all S -stable sets. The statement now follows from the fact that \hat{S} satisfies SSP and S -exclusivity. \square

For example, TC is the finest tournament solution satisfying SSP and CNL -exclusivity, MC is the finest tournament solution satisfying SSP and UC -exclusivity, and ME is the finest tournament solution satisfying SSP and BA -exclusivity.

4.3 The Minimal Extending Set

As mentioned in Section 4.1, the minimal extending set is a new tournament solution that has not been considered before. In analogy to UC -stable sets, which are known as covering sets, we will call BA -stable sets *extending sets*. B is an extending set of tournament $T = (A, \succ)$ if, for any $a \notin B$, every transitive path (or so-called Banks trajectory) in $B \cup \{a\}$ with maximal element a can be extended, i.e., there is $b \in B$ such that b dominates every element on the path. In other words, $B \subseteq A$ is an *extending set* if for all $a \in A \setminus B$, $a \notin BA(B \cup \{a\})$.

If Conjecture 1 is correct, ME satisfies all properties defined in Section 2.5 and is a refinement of BA due to Propositions 4 and 5 and Theorem 5. Proposition 3 furthermore

⁷ UC -exclusivity is the property γ^{**} used in the axiomatization of MC (Laslier, 1997).

implies that, assuming that Conjecture 1 holds, ME is a refinement of MC since every covering set is also an extending set. We refer to Figure 1 for an example tournament T where $ME(T)$ happens to be *strictly* contained in $MC(T)$.⁸

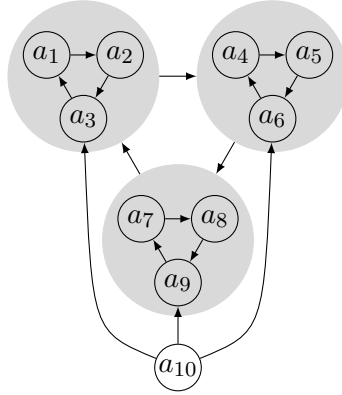


Figure 1: Example tournament $T = (A, \succ)$ where MC and ME differ ($MC(T) = A$ and $ME(T) = A \setminus \{a_{10}\}$). a_{10} only dominates a_3 , a_6 , and a_9 .

A remarkable property of ME is that, just like BA , it is capable of ruling out alternatives in *regular* tournaments, i.e., it satisfies IRR. No irregular tournament solution is known to satisfy all four basic properties. However, if Conjecture 1 were true, ME would be such a concept.

5 Retentiveness and Stability

Motivated by cooperative majority voting, Schwartz (1990) introduced a tournament solution based on a notion he calls *retentiveness*. It turns out that retentiveness bears some similarities to stability. For example, the top cycle can be represented as a minimal stable set as well as a minimal retentive set, albeit using different underlying tournament solutions.

5.1 The Tournament Equilibrium Set

The intuition underlying retentive sets is that alternative a is only “properly” dominated by alternative b if b is chosen among a ’s dominators by some underlying tournament solution S . A set of alternatives is then called S -retentive if none of its elements is properly dominated by some outside alternative with respect to S .

Definition 18. Let S be a tournament solution and $T = (A, \succ)$ a tournament. Then, $B \subseteq A$ is *retentive* in T with respect to tournament solution S (or *S -retentive*) if $B \neq \emptyset$ and $S(\overline{D}(b)) \subseteq B$ for all $b \in B$. The set of S -retentive sets for a given tournament $T = (A, \succ)$ will be denoted by $\mathcal{R}_S(T) = \{B \subseteq A \mid B \text{ is } S\text{-retentive in } T\}$.

⁸This is also the case for a tournament given by Dutta (1990).

S -retentive sets are guaranteed to exist since the set of all alternatives A is trivially S -retentive in (A, \succ) for any tournament solution S . In analogy to Definition 16, the union of minimal S -retentive sets defines a tournament solution.

Definition 19. Let S be a tournament solution. Then, the tournament solution \mathring{S} is defined as

$$\mathring{S}(T) = \bigcup_{\subseteq} \min(\mathcal{R}_S(T)).$$

It is easily verified that \mathring{S} is well-defined as a tournament solution as there are no S -retentive sets that do not contain the Condorcet winner whenever one exists.

As an example, consider the tournament solution $S_{\mathcal{M}_1}$ that always returns all alternatives, i.e., $S_{\mathcal{M}_1}((A, \succ)) = A$. The unique minimal retentive set with respect to $S_{\mathcal{M}_1}$ is the top cycle, that is $TC = \widehat{S}_{\mathcal{M}_2} = \mathring{S}_{\mathcal{M}_1}$.

Schwartz introduced retentiveness in order to recursively define the *tournament equilibrium set* (TEQ) as the union of minimal TEQ -retentive sets. This recursion is well-defined because the order of the dominator set of any alternative is strictly smaller than the order of the original tournament.

Definition 20 (Schwartz, 1990). The *tournament equilibrium set* (TEQ) of a tournament T is defined recursively as $TEQ(T) = \mathring{TEQ}(T)$.

In other words, TEQ is the unique fixed point of the \circ -operator.

Schwartz conjectured that every tournament admits a *unique* minimal TEQ -retentive set. Despite several attempts to prove or disprove this statement (e.g., Laffond et al., 1993a; Houy, 2009), the statement has remained a conjecture. A recent computer analysis failed to find a counter-example in all tournaments of order 12 or less and a fairly large number of random tournaments (Brandt et al., 2009).

Conjecture 2 (Schwartz, 1990). \mathcal{R}_{TEQ} is directed.

The following statement is easily appreciated.

Proposition 8. Let S be an arbitrary tournament solution. Then, the union of two S -retentive sets and the non-empty intersection of two S -retentive sets is also S -retentive.

Proof. Let $T = (A, \succ)$ be a tournament and $B_1, B_2 \in \mathcal{R}_S(T)$. $S(\overline{D}(a)) \subseteq B_1$ for all $a \in B_1$ and $S(\overline{D}(a)) \subseteq B_2$ for all $a \in B_2$. Consequently, $S(\overline{D}(a)) \subseteq B_1 \cup B_2$ for all $a \in B_1 \cup B_2$ and hence $B_1 \cup B_2 \in \mathcal{R}_S(T)$. Now, let $B_1 \cap B_2 \neq \emptyset$ and $a \in B_1 \cap B_2$. Retentiveness of B_1 and B_2 implies that $S(\overline{D}(a)) \subseteq B_1$ and $S(\overline{D}(a)) \subseteq B_2$. Hence, $S(\overline{D}(a)) \subseteq B_1 \cap B_2$ and $B_1 \cap B_2 \in \mathcal{R}_S(T)$. \square

It follows from Proposition 8 that Conjecture 2 is equivalent to the statement that there are no two disjoint TEQ -retentive sets in any tournament.

In Section 3.2, the Banks set was characterized as the finest tournament solution satisfying DCON. It turns out that, if \mathcal{R}_{TEQ} is directed, TEQ is the finest tournament solution satisfying a very natural weakening of DCON, where the dominion contraction property is only required to hold for alternatives contained in the choice set.

Definition 21. A tournament solution S satisfies *weak dominion contraction* (WDCON) if $S(A \setminus (D_{\succeq}(a))) \subseteq S(A)$ for all tournaments $T = (A, \succ)$ and $a \in S(A)$.

Since $S_{\mathcal{Q}}$ satisfies DCON for any concept of qualified subsets \mathcal{Q} , $\widehat{S}_{\mathcal{Q}}$ satisfies WDCON if $\mathcal{S}_{S_{\mathcal{Q}}}$ is directed.

Proposition 9. *Let \mathcal{Q} be a concept of qualified subsets such that $\mathcal{S}_{S_{\mathcal{Q}}}$ is directed. Then, $\widehat{S}_{\mathcal{Q}}$ satisfies WDCON.*

Proof. Let $T = (A, \succ)$ be a tournament, $B = \widehat{S}_{\mathcal{Q}}(T)$, $b \in B$, and $C = B \setminus D_{\succeq}(b)$. We show that C is $S_{\mathcal{Q}}$ -stable in $A \setminus D_{\succeq}(b)$. Let $a \in A \setminus (B \cup D_{\succeq}(b))$ and consider the tournament restricted to $C \cup \{a\}$. We know from B 's stability that $S_{\mathcal{Q}}(B \cup \{a\}) \subseteq B$. Furthermore, since $S_{\mathcal{Q}}$ satisfies DCON (Lemma 1), $S_{\mathcal{Q}}(C \cup \{a\}) \subseteq C$, which shows that C is $S_{\mathcal{Q}}$ -stable and thus contains the minimal $S_{\mathcal{Q}}$ -stable set. \square

Consequently, TC and MC satisfy WDCON and ME satisfies WDCON if Conjecture 1 holds. Some insight reveals that WDCON is strongly related to retentiveness.

Lemma 3. *A tournament solution S satisfies WDCON if and only if $S(T)$ is S -retentive for any tournament T .*

Proof. Let $T = (A, \succ)$ be a tournament. WDCON demands that $S(A \setminus D_{\succeq}(a)) \subseteq S(A)$ for any $a \in S(A)$. Since $A \setminus D_{\succeq}(a) = \overline{D}(a)$, this is equivalent to $S(\overline{D}(a)) \subseteq S(A)$ for any $a \in S(A)$, which precisely characterizes S -retentiveness of $S(A)$. \square

Together with Proposition 9 we thus have that every $S_{\mathcal{Q}}$ -stable set is $\widehat{S}_{\mathcal{Q}}$ -retentive, i.e., $\mathcal{S}_{S_{\mathcal{Q}}} \subseteq \mathcal{R}_{\widehat{S}_{\mathcal{Q}}}$.

Unfortunately, and somewhat surprisingly, it is not known whether TEQ satisfies *any* of the basic properties defined in Section 2.5. However, Laffond et al. (1993a) have shown that if \mathcal{R}_{TEQ} is directed, then TEQ satisfies all basic properties and is furthermore strictly contained in the minimal covering set MC . We show that, given that Conjecture 2 is true, TEQ is the finest tournament solution satisfying WDCON and thus a refinement of *all* tournament solutions $S_{\mathcal{Q}}$ where \mathcal{Q} is a concept of qualified subsets.

Theorem 6. *If \mathcal{R}_{TEQ} is directed, then TEQ is the finest tournament solution satisfying WDCON.*

Proof. Let S be a tournament solution that satisfies WDCON. We prove by induction on the tournament order n that $S(T)$ is TEQ -retentive for any tournament T . The basis is straightforward. For the induction step we may assume that $S(T)$ is TEQ -retentive for all tournaments T of order n or less. Now, consider a tournament T of order $n + 1$ and let $a \in S(T)$. Since $\overline{D}(a)$ contains at most n alternatives, $S(\overline{D}(a))$ is known to be TEQ -retentive and thus contains the unique minimal TEQ -retentive set in $\overline{D}(a)$ (the existence of which follows from the directness of \mathcal{R}_{TEQ}). Hence, $S(T)$ is TEQ -retentive and consequently contains $TEQ(T)$. The fact that TEQ satisfies WDCON follows straightforwardly from Lemma 3. \square

Corollary 2. *Let \mathcal{Q} be a concept of qualified subsets such that $\mathcal{S}_{\mathcal{Q}}$ is directed and assume that \mathcal{R}_{TEQ} is directed. Then, $TEQ \subseteq \hat{\mathcal{S}}_{\mathcal{Q}}$.*

Proof. The statement follows directly from Theorem 6 and Proposition 9. □

In particular, we have $TEQ \subseteq ME$ if \mathcal{R}_{TEQ} is directed. This leaves open the question of whether TEQ and ME are actually different solution concepts. The tournament given in Figure 2 demonstrates that this is indeed the case.

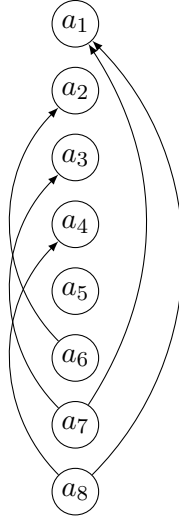


Figure 2: Example tournament $T = (A, \succ)$ where ME and TEQ differ ($ME(T) = A$ and $TEQ(T) = A \setminus \{a_5\}$). In particular, $A \setminus \{a_5\}$ is no extending set since $a_5 \in BA(A)$ via the non-extendable transitive set $\{a_5, a_6, a_7, a_8\}$.

5.2 TEQ as a Minimal Stable Set

A natural question is whether TEQ itself can be represented as a minimal stable set. The following two lemmas establish that this is indeed the case if \mathcal{R}_{TEQ} is directed. We first show that every S -retentive set is \hat{S} -stable if S satisfies WSP.

Lemma 4. *Let S be a tournament solution that satisfies WSP. Then, $\mathcal{R}_S \subseteq \mathcal{S}_{\hat{S}}$.*

Proof. Let $T = (A, \succ)$ be a tournament and B an S -retentive set in A . If $B = A$, the statement is trivially satisfied. Otherwise, let $a \in A \setminus B$. We first show that B is S -retentive in $B \cup \{a\}$. Let b be an arbitrary alternative in B . S -retentiveness implies $S(\overline{D}(b)) \subseteq B$ and WSP implies $S(\overline{D}_{B \cup \{a\}}(b)) \subseteq S(\overline{D}(b))$. As a consequence, $S(\overline{D}_{B \cup \{a\}}(b)) \subseteq B$, and thus B is S -retentive in $B \cup \{a\}$. It remains to be shown that a is not contained in another minimal S -retentive subset of $B \cup \{a\}$. Assume for contradiction that a is contained in some minimal S -retentive set. If $\{a\}$ itself were an S -retentive set, a would be the Condorcet winner in $B \cup \{a\}$, contradicting the fact that

B is S -retentive in $B \cup \{a\}$. Now let $C \subset B \cup \{a\}$ with $a \in C$ and $|C| > 1$ be a minimal S -retentive set. According to Proposition 8, $B \cap C$ is also S -retentive, contradicting the minimality of C . It follows that B is \dot{S} -stable. \square

Assuming the directedness of \mathcal{R}_{TEQ} , it can be shown that every TEQ -stable set is also TEQ -retentive.

Lemma 5. *If \mathcal{R}_{TEQ} is directed, then $\mathcal{S}_{TEQ} \subseteq \mathcal{R}_{TEQ}$.*

Proof. Let $T = (A, \succ)$ be a tournament. It is sufficient to prove the following statement: Let $B \subseteq A$, $b_1, b_2 \in A \setminus B$, $C_1 = TEQ(B \cup \{b_1\})$, and $C_2 = TEQ(B \cup \{b_2\})$ such that $b_1 \notin C_1$ and $b_2 \notin C_2$. Then $C_1 = C_2 = C$ and C is TEQ -retentive in $B \cup \{b_1, b_2\}$.

We may assume that $b_1 \neq b_2$ since the statement is trivially satisfied if $b_1 = b_2$. We prove the statement by induction on the size of B . For the basis, let $B = \{b\}$. We then have $b \succ b_1$ and $b \succ b_2$, which makes b a Condorcet winner in $\{b, b_1, b_2\}$ and hence a TEQ -retentive subset of $\{b, b_1, b_2\}$. Now assume that the statement holds for any set of size up to k . Let $|B| = k + 1$ and b_1, b_2, C_1 , and C_2 be defined as in the statement. Directedness of \mathcal{R}_{TEQ} is equivalent to TEQ satisfying SSP (Laffond et al., 1993a). Since $b_1 \neq b_2$, SSP implies that $C_1 = C_2 = C$. Let $a \in C$, $C'_1 = TEQ(\overline{D}_{B \cup \{b_1\}}(a))$, and $C'_2 = TEQ(\overline{D}_{B \cup \{b_2\}}(a))$. If $b_1 \notin \overline{D}(a)$ or $b_2 \notin \overline{D}(a)$, $TEQ(\overline{D}_{B \cup \{b_1, b_2\}}(a)) \subseteq C'_1 = C'_2 = C'$ since, according to Proposition 8, C'_1 is TEQ -retentive in $\overline{D}_{B \cup \{b_1\}}(a)$ and C'_2 is TEQ -retentive in $\overline{D}_{B \cup \{b_2\}}(a)$. Otherwise we can apply the induction hypothesis to $\overline{D}_B(a)$, since $b_1 \notin C'_1$ and $b_2 \notin C'_2$ due to the fact that $a \in C$. Thus, $C'_1 = C'_2 = C'$ is TEQ -retentive in $\overline{D}_{B \cup \{b_1, b_2\}}(a)$. Conjecture 2 further implies that $TEQ(\overline{D}_{B \cup \{b_1, b_2\}}(a)) \subseteq C'$. As a consequence, C is TEQ -retentive in $B \cup \{b_1, b_2\}$. \square

We have now cleared the ground for the main result of this section.

Theorem 7. *If \mathcal{R}_{TEQ} is directed, then $TEQ = \widehat{TEQ}$.*

Proof. Since $TEQ = \dot{TEQ}$ by definition, Lemma 4 implies that every TEQ -retentive set is TEQ -stable, given that TEQ satisfies WSP. Assuming that \mathcal{R}_{TEQ} is directed, a simple inductive argument shows that TEQ indeed satisfies WSP. Lemma 5, on the other hand, establishes that every TEQ -stable set is also TEQ -retentive, which completes the proof as $\mathcal{R}_{TEQ} = \mathcal{S}_{TEQ}$. \square

Combining Definition 20 and Theorem 7 and assuming that Conjecture 2 is true, we obtain the appealing equation

$$TEQ = \dot{TEQ} = \widehat{TEQ}.$$

While TEQ is the only tournament solution S such that $S = \dot{S}$, we know from Theorem 5 that all tournament solutions $S = \widehat{S}_{\mathcal{Q}}$ such that $\mathcal{S}_{\mathcal{Q}}$ is directed satisfy $S = \widehat{S}$. The equation above raises the question whether there are other tournament solutions that are their own minimal stable and minimal retentive set, i.e., tournament solutions S such that $\dot{S} = \widehat{S}$. It can be shown that TEQ is the finest “reasonable” concept among these. However, whether any such tournament solution besides TEQ exists remains unknown.

Proposition 10. *Let S be a tournament solution that satisfies WSP, $\widehat{S} = \mathring{S}$, and \mathcal{S}_S is directed. Then, $TEQ \subseteq S$ if \mathcal{R}_{TEQ} is directed.*

Proof. Let $T = (A, \succ)$ be a tournament. We prove by induction on the tournament order n that every S -retentive set $B \subseteq A$ is also TEQ -retentive if $\widehat{S} = \mathring{S}$ and S satisfies WSP. The basis is straightforward. For the induction step consider a tournament of order $n + 1$ that contains an S -retentive set B and let $b \in B$. We know from the induction hypothesis that $TEQ(\overline{D}(b)) \subseteq \mathring{S}(\overline{D}(b)) = \widehat{S}(\overline{D}(b))$. Since S satisfies WSP and \mathcal{S}_S is directed, $\widehat{S}(\overline{D}(b)) \subseteq S(\overline{D}(b)) \subseteq B$. Hence, $TEQ(\overline{D}(b)) \subseteq B$ and B is TEQ -retentive. \square

Theorem 7 entails that Conjecture 2 is at least as strong as Conjecture 1.

Theorem 8. *If \mathcal{R}_{TEQ} is directed, then $\mathcal{S}_{\mathcal{M}^*}$ is directed.*

Proof. We prove the statement by contradiction. Assume that \mathcal{R}_{TEQ} is directed, but $\mathcal{S}_{\mathcal{M}^*}$ is not. The directedness of \mathcal{R}_{TEQ} implies the directedness of \mathcal{S}_{TEQ} (Theorem 7). Furthermore, Corollary 2 states that TEQ is a refinement of $\widehat{S}_{\mathcal{Q}}$ for any concept of qualified subsets \mathcal{Q} . Hence, every $\mathcal{S}_{\mathcal{M}^*}$ -stable set is also TEQ -stable. According to Theorem 2, $\mathcal{S}_{\mathcal{M}^*}$ is *not* directed if and only if there are two disjoint $\mathcal{S}_{\mathcal{M}^*}$ -stable sets. However, these sets would also constitute two disjoint TEQ -stable sets, a contradiction. \square

Similarly, the (known) directedness of $\mathcal{S}_{\mathcal{M}}$ (cf. Theorem 3) also follows from the conjectured directedness of \mathcal{R}_{TEQ} . We finally show that Conjecture 2 is equivalent to the statement that there are no two disjoint TEQ -stable sets in any tournament.

Theorem 9. *\mathcal{R}_{TEQ} is directed if and only if \mathcal{S}_{TEQ} pairwise intersects.*

Proof. The direction from left to right follows from Theorem 7. The converse implication is proven by induction on the tournament order n . The basis is straightforward. For the induction step, we may assume that \mathcal{R}_{TEQ} is directed for all tournaments of order n or less. A simple inductive argument shows that TEQ satisfies WSP in these tournaments. The proof of Lemma 4 actually shows that, given that TEQ meets WSP for all tournaments of order at most n , $\mathcal{R}_{TEQ} \subseteq \mathcal{S}_{TEQ}$ for all tournaments with up to $n + 1$ alternatives. Now assume for contradiction that there is a tournament of order $n + 1$ that contains two disjoint TEQ -retentive sets. It follows from Lemma 4 that these sets are also TEQ -stable, a contradiction. \square

6 Quantitative Concepts

In Section 3, several solution concepts were defined by collecting the maximal elements of inclusion-maximal qualified subsets. In this section, we replace maximality with respect to set inclusion by maximality with respect to *cardinality*, i.e., we look at qualified subsets containing the largest number of elements.

6.1 Maximal Qualified Subsets

For any set of finite sets \mathcal{S} , define $\max_{\leq}(\mathcal{S}) = \{S \in \mathcal{S} \mid |S| \geq |S'| \text{ for all } S' \in \mathcal{S}\}$. In analogy to Definition 13, we can now define a solution concept that yields the maximal elements of the largest qualified subsets.

Definition 22. Let \mathcal{Q} be a concept of qualified subsets. Then, the tournament solution $S_{\mathcal{Q}}^{\#}$ is defined as

$$S_{\mathcal{Q}}^{\#}(T) = \{\max_{\leq}(B) \mid B \in \max_{\leq}(\mathcal{Q}(T))\}.$$

Obviously, $S_{\mathcal{Q}}^{\#}(T) \subseteq S_{\mathcal{Q}}(T)$ for any concept of qualified subsets \mathcal{Q} . For the concept of qualified subsets \mathcal{M} , i.e., the set of subsets that admit a maximal element, we obtain the Copeland set.

Copeland set. $S_{\mathcal{M}}^{\#}(T)$ returns the *Copeland set* $CO(T)$ of a tournament T , i.e., the set of all alternatives whose dominion is of maximal size (Copeland, 1951).⁹

6.2 Minimal Stable Sets

When taking the Copeland set as the basis for minimal stable sets, some tournaments contain more than one inclusion-minimal externally stable set and, even worse, do not admit a set that satisfies both internal *and* external stability (see Figure 3).

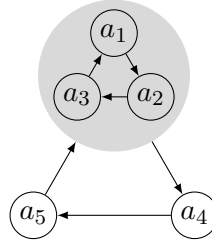


Figure 3: Example tournament $T = (A, \succ)$ that does not contain an internally and externally CO -stable set. There are four externally CO -stable sets (A , $\{a_2, a_3, a_4, a_5\}$, $\{a_1, a_3, a_4, a_5\}$, and $\{a_1, a_2, a_4, a_5\}$), none of which is internally stable.

However, as it turns out, every tournament is the *summary* of some tournament consisting only of homogeneous components that admits a unique internally and externally CO -stable set. For example, when replacing a_4 and a_5 in the tournament given in Figure 3 with 3-cycle components, the set of all alternatives is internally and externally stable. The following definition captures this strengthened notion of stability.

⁹This set is usually attributed to Copeland despite the fact that Zermelo (1929) and Lull (as early as 1283, see Hägele and Pukelsheim 2001) have suggested equivalent concepts much earlier.

Definition 23. Let S be a tournament solution and $\tilde{T} = (\{1, \dots, k\}, \succsim)$ a tournament. Then, $\tilde{B} \subseteq \{1, \dots, k\}$ is *strongly stable* with respect to tournament solution S (or *strongly S -stable*) if there exist homogeneous tournaments T_1, \dots, T_k on k disjoint sets $B_1, \dots, B_k \subseteq X$ such that $B = \bigcup_{i \in \tilde{B}} B_i$ is internally and externally S -stable in $T = \Pi(\tilde{T}, T_1, \dots, T_k)$. The set of strongly S -stable sets for a given tournament $T = (A, \succ)$ will be denoted by $\tilde{\mathcal{S}}_S(T) = \{B \subseteq A \mid B \text{ is strongly } S\text{-stable in } T\}$.

Now, in analogy to \hat{S} , we can define a tournament solution that yields the minimal strongly S -stable set with respect to some underlying tournament solution S .

Definition 24. Let S be a tournament solution. Then,

$$\hat{S}(T) = \bigcup_{\subseteq} \min(\tilde{\mathcal{S}}_S(T)).$$

The following result follows straightforwardly from observations made independently by Laffond et al. (1993b) and Fisher and Ryan (1995) (see, also Laslier, 1997, 2000).

Theorem 10 (Laffond et al., 1993b). $|\tilde{\mathcal{S}}_{S_{\mathcal{M}}^{\#}}(T)| = 1$ for any tournament T .

The unique strongly stable set with respect to the Copeland set is the *bipartisan set*.

Bipartisan set. The *bipartisan set* of a tournament T is given by $BP(T) = \hat{S}_{\mathcal{M}}^{\#}(T)$, i.e., the set of alternatives corresponding to the support of the unique Nash equilibrium of the tournament game (Laffond et al., 1993b).

BP satisfies all basic properties, composition-consistency, and is contained in MC . Its relationship with BA is unknown (Laffond et al., 1993b).

Interestingly, minimality is not required for $\hat{S}_{\mathcal{M}}^{\#}$, because there can be at most one $S_{\mathcal{M}}^{\#}$ -strongly stable set. Please observe that stable sets and strongly stable sets coincide for tournament solutions based on minimal stable sets that satisfy COM and are internally stable, such as MC , ME , and TEQ . Even though, TC does not satisfy COM, it is easily verified that replacing alternatives with homogeneous components does not affect the top cycle of a tournament. It is thus possible to define all mentioned concepts using minimal strongly stable sets instead of stable sets (see Table 1).

S	\hat{S}
Condorcet non-losers (CNL)	Top cycle (TC)
Copeland set (CO)	Bipartisan set (BP)
Uncovered set (UC)	Minimal covering set (MC)
Banks set (BA)	Minimal extending set (ME)
Tournament equilibrium set (TEQ)	Tournament equilibrium set (TEQ)

Table 1: Tournament solutions and their minimal strongly stable counterparts. The representation of TEQ as a stable set relies on Conjecture 2.

Solution Concept	Origin	MON	IUA	WSP	SSP	COM	IRR	EFF
$S_{\mathcal{M}_2}$ (<i>CNL</i>)		✓	✓	✓	–	–	–	✓
$S_{\mathcal{M}}$ (<i>UC</i>)	Fishburn (1977); Miller (1980)	✓	–	✓	–	✓	–	✓
$S_{\mathcal{M}^*}$ (<i>BA</i>)	Banks (1985)	✓	–	✓	–	✓	✓	– ^a
$\hat{S}_{\mathcal{M}_2}$ (<i>TC</i>)	Good (1971); Smith (1973)	✓	✓	✓	✓	–	–	✓
$\hat{S}_{\mathcal{M}}$ (<i>MC</i>)	Dutta (1988)	✓	✓	✓	✓	✓	–	✓
$\hat{S}_{\mathcal{M}^*}$ (<i>ME</i>)	Brandt (2008)	✓ ^b	✓ ^b	✓ ^b	✓ ^b	✓	✓	– ^{a,b}
\widehat{TEQ} (<i>TEQ</i>)	Schwartz (1990)	✓ ^c	✓ ^c	✓ ^c	✓ ^c	✓	✓	– ^a
$S_{\mathcal{M}}^{\#}$ (<i>CO</i>)	Copeland (1951)	✓	–	–	–	–	–	✓
$\hat{S}_{\mathcal{M}}^{\#}$ (<i>BP</i>)	Laffond et al. (1993b)	✓	✓	✓	✓	✓	–	✓

^aThere exists no efficient algorithm unless P equals NP.

^bThis statement relies on Conjecture 1.

^cThis statement relies on Conjecture 2.

Table 2: Properties of solution concepts (MON: monotonicity, IUA: independence of unchosen alternatives, WSP: weak superset property, SSP: strong superset property, COM: composition-consistency, IRR: irregularity, EFF: efficient computability (the tournament solution can be computed in polynomial time))

7 Conclusion

We provided a unifying treatment of tournament solutions based on maximal qualified subsets and minimal stable sets. Given the results of Sections 4 and 5, a central role in the theory of tournament solutions may be ascribed to Conjecture 2, which states that no tournament contains two disjoint *TEQ*-retentive sets. Conjecture 2 (or the equivalent statement that there are no two disjoint *TEQ*-stable sets) has a number of appealing consequences on minimal stable sets:

- Every tournament T admits a unique *minimal dominating set* $TC(T)$ (as shown by Good, 1971). Consequently, TC satisfies all basic properties and is the finest solution concept satisfying SSP and *CNL*-exclusivity.
- Every tournament T admits a unique *minimal covering set* $MC(T)$ (as shown by Dutta, 1988). Consequently, MC satisfies all basic properties and is the finest solution concept satisfying SSP and *UC*-exclusivity.
- Every tournament T admits a unique *minimal extending set* $ME(T)$ (open problem). Consequently, ME satisfies all basic properties and is the finest solution concept satisfying SSP and *BA*-exclusivity.
- Every tournament T admits a unique *minimal TEQ-retentive set* $TEQ(T)$ (open problem). Consequently, TEQ satisfies all basic properties and is the finest solution concept satisfying WDCON and the finest solution concept S such that S satisfies SSP and, for any tournament T , $S(T) = A \setminus \{a\}$ implies $a \notin S(\overline{D}(b))$ for any $b \in A$. Moreover, TEQ is the finest solution concept S satisfying WSP and $\hat{S} = \hat{S}$.

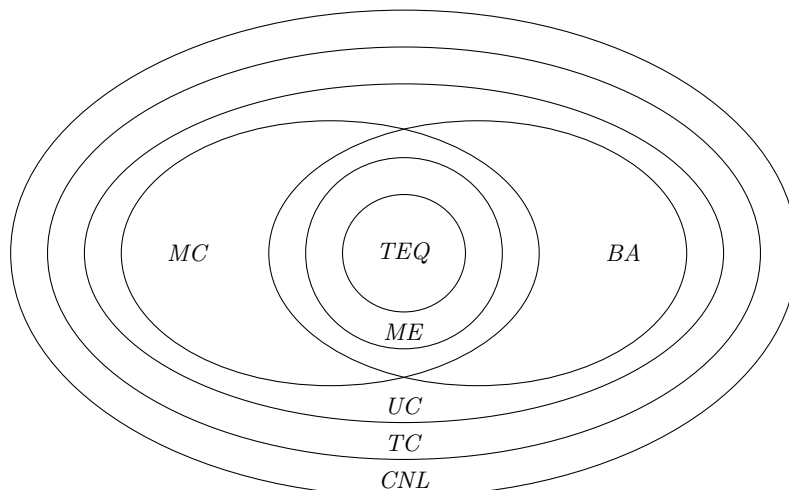


Figure 4: Set-theoretic relationships between qualitative tournament solutions. BA and MC are not included in each other, but they always intersect (see, e.g., Laslier, 1997). The inclusion of TEQ in ME relies on Conjecture 2 and that of ME in MC on Conjecture 1 (which is implied by Conjecture 2). CO is contained in UC but may be disjoint from MC and BA . The exact location of BP in this diagram is unknown (BP is contained in MC and intersects with TEQ in all known instances (Laslier, 1997)).

- $TEQ \subseteq ME \subseteq MC \subseteq TC$ and $ME \subseteq BA$.¹⁰

Conjecture 1 is a weaker version of Conjecture 2, which implies all of the above statements except those that involve TEQ .

Table 2 and Figure 4 summarize the properties and set-theoretic relationships of the most important considered tournament solutions, respectively.

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¹⁰A consequence of these inclusions is that deciding whether an alternative is contained in the minimal extending set of a tournament is NP-hard. This follows from a proof by Brandt et al. (2009), which establishes hardness of all solution concepts that are sandwiched between BA and TEQ .

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