Deciding Membership in Minimal Upward Covering Sets is Hard for Parallel Access to NP*

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Abstract

A common thread in the social sciences is to identify the "most desirable" elements of a set of alternatives according to some binary dominance relation. Examples can be found in areas as diverse as voting theory, game theory, and argumentation theory. Brandt and Fischer [BF08] proved that $\mathrm{MC_u}$-MEMBER, the problem of deciding whether an alternative is contained in some inclusion-minimal upward covering set—a subset of alternatives that satisfies certain notions of internal and external stability—is NP-hard. We raise their NP-hardness lower bound to the Θ_2^p level of the polynomial hierarchy and provide a Σ_2^p upper bound. Relatedly, we show that other problems regarding minimal upward covering sets, such as deciding whether the set of all alternatives is a minimal upward covering set, are coNP-hard. As a consequence, minimal upward covering sets cannot be found in polynomial time unless $\mathrm{P} = \mathrm{NP}$.

1 Introduction

A common thread in the social sciences is to identify the "most desirable" elements of a set of alternatives according to some binary dominance relation. Applications range from cooperative to non-cooperative game theory, from social choice theory to argumentation theory, and from multi-criteria decision analysis to sports tournaments (see, e.g., [Las97, BF08] and the references therein).

In social choice settings, the most common dominance relation is the pairwise majority relation, where an alternative x is said to dominate another alternative y if the number of individuals preferring x to y exceeds the number of individuals preferring y to x. McGarvey [McG53] proved that *every* asymmetric dominance relation can be realized via a particular preference profile, even if the individual preferences are linear. For example, Condorcet's well-known paradox says that the majority relation may contain cycles and thus does not always have maximal elements, even if all the underlying individual preferences do. This means that the concept of

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maximality is rendered useless in many cases, which is why various so-called *solution concepts* have been proposed. Solution concepts can be used in place of maximality for nontransitive relations (see, e.g., [Las97]). In particular, concepts based on so-called *covering relations*—transitive subrelations of the dominance relation at hand—have turned out to be very attractive [Fis77, Mil80, Dut88].

In this paper, we will be concerned with the notion of *upward covering* [Bor83], where an alternative x is said to upward cover another alternative y if x dominates y and every alternative dominating x also dominates y. The intuition is that x "strongly" dominates y in the sense that there is no alternative that dominates x but not y. A *minimal upward covering set* is defined as an inclusion-minimal set of alternatives that satisfies certain notions of internal and external stability with respect to the upward covering relation [Dut88, BF08].

Recent work in theoretical computer science has addressed the computational complexity of most solution concepts proposed in the context of binary dominance (see, e.g., [Woe03, Alo06, Con06, BFH07, BF08, BFHM08]). In particular, Brandt and Fischer [BF08] have shown that $\mathrm{MC_u}$-MEMBER, the problem of deciding whether an alternative is contained in some minimal upward covering set, is NP-hard. We improve on this result by raising their NP-hardness lower bound to the Θ_2^p level of the polynomial hierarchy and provide an upper bound of Σ_2^p. On the way, we prove that other problems related to minimal upward covering sets are coNP-hard. This implies that minimal upward covering sets cannot be found in polynomial time unless $\mathrm{P} = \mathrm{NP}$.

2 Definitions and Notation

In this section, we define the required notions and notation from social choice theory and complexity theory.

Definition 2.1 (Upward Covering Relation) *Let A be a finite set of alternatives, let $\succ \subseteq A \times A$ be a dominance relation on A, i.e., $\succ$ is asymmetric and irreflexive.*[1] *A dominance relation $\succ$ on a set A of alternatives can be conveniently represented as a* dominance graph, *denoted by $(A, \succ)$, whose vertices are the alternatives from A, and for each $x, y \in A$ there is a directed edge from x to y if and only if $x \succ y$.*

For any two alternatives x and y in A, define the following covering relation (see, e.g., [Fis77, Mil80, Bor83]): x upward covers *y, denoted by $x C_u y$, if $x \succ y$ and for all $z \in A$, $z \succ x$ implies $z \succ y$.*

Definition 2.2 (Upward Uncovered Set) *Let A be a set of alternatives, let $B \subseteq A$ be any subset, let $\succ$ be a dominance relation on A, and let C_u be the upward covering relation on A based on $\succ$. The* upward uncovered set of B with respect to C *is defined as*

$$\mathrm{UC}_u(B) = \{x \in B \mid y C_u x \text{ for no } y \in B\}.$$

For the upward covering relation, transitivity of the relation implies nonemptiness of the upward uncovered set for each nonempty set of alternatives. Every upward uncovered set contains one or more *minimal upward covering sets* [BF08]. Dutta [Dut88] proposed minimal covering sets in the context of tournaments, i.e., complete dominance relations. Minimal upward covering sets are one of several possible generalizations to incomplete dominance relations (for more details, see [BF08]). The intuition underlying covering sets is that there should be no reason to restrict the selection by excluding some alternative from it (internal stability) and there should be an argument against each proposal to include an outside alternative into the selection (external stability).

Definition 2.3 (Minimal Upward Covering Set) *Let A be a set of alternatives, on which a dominance relation*

[1] In general, $\succ$ need not be transitive or complete. For alternatives x and y, $x \succ y$ (equivalently, $(x, y) \in \succ$) is interpreted as x being strictly preferred to y (and we say "x dominates y"), for example as the result of a strict majority of voters preferring x to y.

and the corresponding upward covering relation are defined. A subset $B \subseteq A$ is an upward covering set for *A if the following two properties hold:*

- Internal stability: $\mathrm{UC}_u(B) = B$.
- External stability: *For all $x \in A - B$, $x \notin \mathrm{UC}_u(B \cup \{x\})$.*

An upward covering set M for A is said to be (inclusion-)minimal *if no $M' \subset M$ is an upward covering set for A.*

Occasionally, it might be helpful to specify the dominance relation explicitly to avoid ambiguity. In such cases we refer to the dominance graph used and write, e.g., "M is an upward covering set for $(A, \succ)$."

The computational problem of central interest in this paper is $\mathrm{MC_u}$-MEMBER, which is formally defined as follows.

Name: Minimal Upward Covering Set Member ($\mathrm{MC_u}$-MEMBER, for short).
Instance: A set of alternatives A, a dominance relation $\succ$ on A, and a distinguished element $d \in A$.
Question: Is d contained in some minimal upward covering set for A?

We assume that the reader is familiar with the basic notions of complexity theory, such as the polynomial-time many-one reducibility and the related notions of hardness and completeness, and also with standard complexity classes such as P, NP, coNP, and the polynomial hierarchy [MS72] (see also, e.g., the textbooks [Pap94, Rot05]). In particular, coNP is the class of sets whose complements are in NP. $\Sigma_2^p = \mathrm{NP}^{\mathrm{NP}}$, the second level of the polynomial hierarchy, consists of all sets that can be solved by an NP oracle machine that has access (in the sense of a Turing reduction) to an NP oracle set such as SAT. SAT denotes the satisfiability problem of propositional logic, which is one of the standard NP-complete problems (see, e.g., Garey and Johnson [GJ79]) and is defined as follows: Given a boolean formula in conjunctive normal form, does there exist a truth assignment to its variables that satisfies the formula?

Papadimitriou and Zachos [PZ83] introduced the class of problems that can be decided by a P machine that accesses its NP oracle in a parallel manner. This class is also known as the Θ_2^p level of the polynomial hierarchy (see Wagner [Wag90]), and has been shown to coincide with the class of problems solvable in polynomial time via asking $\mathcal{O}(\log n)$ sequential Turing queries to NP (see [Hem87, KSW87]). Equivalently, Θ_2^p is the closure of NP under polynomial-time truth-table reductions. It follows immediately from the definitions that $\mathrm{P} \subseteq \mathrm{NP} \cap \mathrm{coNP} \subseteq \mathrm{NP} \cup \mathrm{coNP} \subseteq \Theta_2^p \subseteq \Sigma_2^p$.

Θ_2^p captures the complexity of various optimization problems. For example, the problem of testing whether the size of a maximum clique in a given graph is an odd number, the problem of deciding whether two given graphs have minimum vertex covers of the same size, and the problem of recognizing those graphs for which certain heuristics yield good approximations for the size of a maximum independent set or for the size of a minimum vertex cover each are known to be complete for Θ_2^p (see [Wag87, HR98, HRS06]). Hemaspaandra and Wechsung [HW02] proved that the minimization problem for boolean formulas is Θ_2^p-hard. In the field of computational social choice, the winner problems for Dodgson [Dod76], Young [You77], and Kemeny [Kem59] elections have been shown to be Θ_2^p-complete in the nonunique-winner model [HHR97, RSV03, HSV05], and also in the unique-winner model [HHR06].

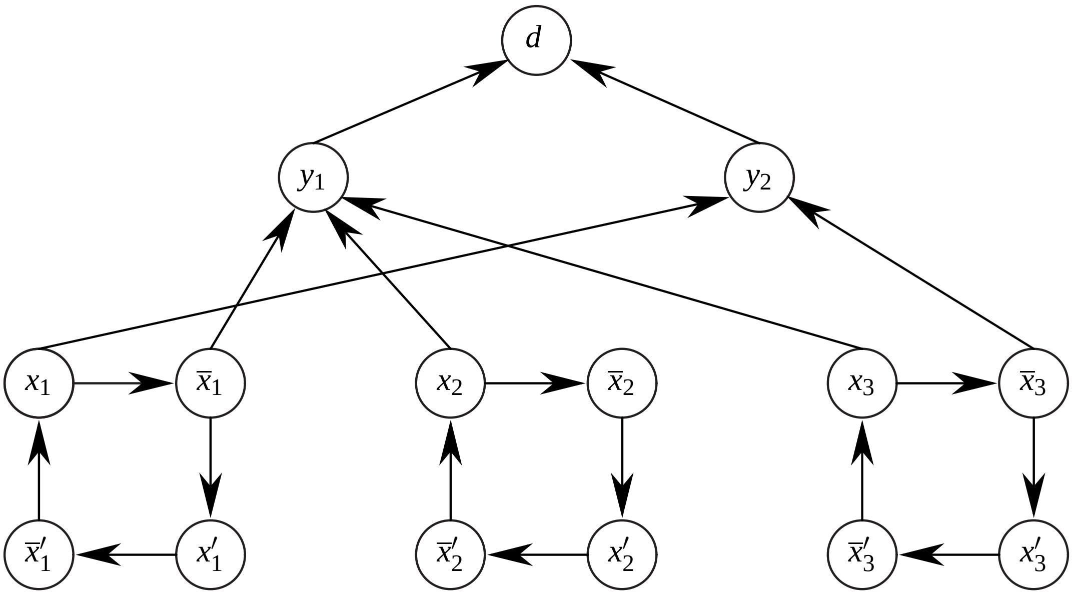


Figure 1: Dominance graph resulting from the formula $(\neg v_1 \vee v_2 \vee v_3) \wedge (v_1 \vee \neg v_3)$ for Theorem 3.1.

3 Main Result

Brandt and Fischer [BF08] proved the following result. Since we will need their reduction in our proof of Theorem 3.2 below, we provide a proof sketch for Theorem 3.1.

Theorem 3.1 ([BF08]) *Deciding whether a designated alternative is contained in some minimal upward covering set for a given dominance graph (i.e.,* $\mathrm{MC_u}$-MEMBER*) is* NP*-hard.*

Proof Sketch. NP-hardness is shown by a reduction from SAT. Given a boolean formula $\varphi(v_1, v_2, \ldots, v_n) = c_1 \wedge c_2 \wedge \cdots \wedge c_r$ over the set $V = \{v_1, v_2, \ldots, v_n\}$ of variables, construct an instance $(A, \succ, d)$ of $\mathrm{MC_u}$-MEMBER as follows. The set of alternatives is

$$A = \{x_i, \overline{x}_i, x'_i, \overline{x}'_i \mid v_i \in V\} \cup \{y_j \mid c_j \text{ is a clause in } \varphi\} \cup \{d\},$$

where d is the distinguished alternative whose membership in a minimal upward covering set for A is to be decided, and the dominance relation $\succ$ is defined by:

- For each i, $1 \le i \le n$, there is a cycle $x_i \succ \overline{x}_i \succ x'_i \succ \overline{x}'_i \succ x_i$;
- if variable v_i occurs in clause c_j as a positive literal, then $x_i \succ y_j$;
- if variable v_i occurs in clause c_j as a negative literal, then $\overline{x}_i \succ y_j$; and
- for each j, $1 \le j \le r$, we have $y_j \succ d$.

As an example of this reduction, Figure 1 shows the dominance graph resulting from the formula $(\neg v_1 \vee v_2 \vee v_3) \wedge (v_1 \vee \neg v_3)$, which is satisfiable, for example via the truth assignment that sets each of v_1, v_2, and v_3 to false. Note that in this case the set $\{\overline{x}_1, \overline{x}'_1, \overline{x}_2, \overline{x}'_2, \overline{x}_3, \overline{x}'_3\}$ is a minimal upward covering set for A, so the instance constructed indeed is in MC_u-MEMBER. In general, Brandt and Fischer [BF08] proved that there exists a satisfying assignment for φ if and only if d is contained in some minimal upward covering set for A. ❑

In Theorem 3.2 below, we raise the NP-hardness lower bound for MC_u-MEMBER, stated in Theorem 3.1, to Θ_2^p-hardness. The proof of Theorem 3.2 is provided in the next section.

Theorem 3.2 *Deciding whether a designated alternative is contained in some minimal upward covering set for a given dominance graph (i.e.,* MC_u-MEMBER*) is* Θ_2^p*-hard.*

It is not very hard to see that Σ_2^p is an upper bound for MC_u-MEMBER.

Proposition 3.3 *Deciding whether a designated alternative is contained in some minimal upward covering set for a given dominance graph (i.e.,* MC_u-MEMBER*) is in* Σ_2^p.

Proof. Let $(A,\succ)$ be a dominance graph and d a designated alternative in A. First, observe that we can verify in polynomial time whether a subset of A is an upward covering set simply by checking whether it satisfies internal and external stability. Now, we can guess an upward covering set $B \subseteq A$ with $d \in B$ in nondeterministic polynomial time and verify its minimality by checking that none of its subsets are upward coverings sets. This places MC_u-MEMBER in NP^{coNP} and consequently in Σ_2^p. ❑

4 Proof of Theorem 3.2

The remainder of this paper is devoted to proving the lower bound for MC_u-MEMBER. We will do so in two steps. Having already stated Brandt and Fischer's NP-hardness result for MC_u-MEMBER [BF08] in Theorem 3.1, we will start by showing, as Lemma 4.1, that MC_u-MEMBER is coNP-hard as well. Then, merging this construction with the construction for NP-hardness from the proof sketch of Theorem 3.1 and extending this appropriately, we will apply Wagner's sufficient condition for proving Θ_2^p-hardness (which is stated as Lemma 4.5 below) to show that MC_u-MEMBER is hard for Θ_2^p.

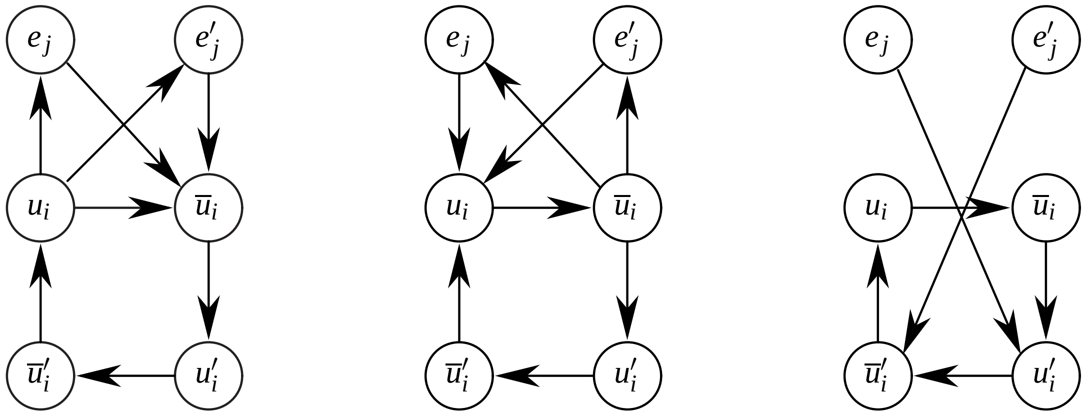
We start by proving that MC_u-MEMBER is coNP-hard.

Lemma 4.1 MC_u-MEMBER *is* coNP*-hard.*

Proof. We provide a reduction from SAT to the complement of MC_u-MEMBER. Given a boolean formula in conjunctive normal form, $\varphi(w_1,w_2,\dots,w_k) = f_1 \wedge f_2 \wedge \cdots \wedge f_\ell$, over the set $W = \{w_1,w_2,\dots,w_k\}$ of variables, we construct an instance $(A,\succ,e_1)$ of MC_u-MEMBER such that φ is satisfiable if and only if e_1 does *not* belong to any minimal upward covering set for A.

The set of alternatives is $A = \{u_i,\overline{u}_i,u'_i,\overline{u}'_i \mid w_i \in W\} \cup \{e_j,e'_j \mid f_j \text{ is a clause in } \varphi\} \cup \{a_1,a_2,a_3\}$, the alternative for which we ask about membership in a minimal upward covering set for A is e_1, and the dominance relation $\succ$ is defined by:

- For each i, $1 \le i \le k$, there is a cycle $u_i \succ \overline{u}_i \succ u'_i \succ \overline{u}'_i \succ u_i$;
- if w_i occurs in f_j as a positive literal, then $u_i \succ e_j$, $u_i \succ e'_j$, $e_j \succ \overline{u}_i$, and $e'_j \succ \overline{u}_i$;
- if w_i occurs in f_j as a negative literal, then $\overline{u}_i \succ e_j$, $\overline{u}_i \succ e'_j$, $e_j \succ u_i$, and $e'_j \succ u_i$;
- if w_i does not occur in f_j, then $e_j \succ u'_i$ and $e'_j \succ \overline{u}'_i$;



(a) w_i occurs in f_j as a positive literal (b) w_i occurs in f_j as a negative literal (c) w_i does not occur in f_j

Figure 2: Parts of the dominance graph constructed for Lemma 4.1.

- for each j, $1 \leq j \leq \ell$, we have $a_1 \succ e_j$ and $a_1 \succ e'_j$; and
- there is a cycle $a_1 \succ a_2 \succ a_3 \succ a_1$.

Figure 2 shows some parts of the dominance graph that results from the given boolean formula φ. In particular, Figure 2(a) shows that part of this graph that corresponds to some variable w_i occuring in clause f_j as a positive literal; Figure 2(b) shows that part of this graph that corresponds to some variable w_i occuring in clause f_j as a negative literal; and Figure 2(c) shows that part of this graph that corresponds to some variable w_i not occuring in clause f_j.

As a more complete example, Figure 3 shows the entire dominance graph that corresponds to the concrete formula $(\neg w_1 \vee w_2) \wedge (w_1 \vee \neg w_3)$, which can be satisfied by setting, for example, each of w_1, w_2, and w_3 to true. A minimal upward covering set corresponding to this assignment is $M = \{u_1, u'_1, u_2, u'_2, u_3, u'_3, a_1, a_2, a_3\}$. Note that our designated alternative e_1 doesn't occur in M, and doesn't occur in any other minimal upward covering set either. This can be seen as follows for the example shown in Figure 3. If there were a minimal upward covering set M' containing e_1 (and thus also e'_1, since they both are dominated by the same alternatives) then neither $\overline{u}_1$ nor u_2 (which dominate e_1) must upward cover e_1, so all alternatives corresponding to the variables w_1 and w_2 (i.e., $\{u_i, \overline{u}_i, u'_i, \overline{u}'_i \mid i \in \{1,2\}\}$) would also have to be contained in M'. Due to $e_1 \succ u'_3$ and $e'_1 \succ \overline{u}'_3$, all alternatives correponding to w_3 (i.e., $\{u_3, \overline{u}_3, u'_3, \overline{u}'_3\}$) are in M' as well. Consequently, e_2 and e'_2 are no longer upward covered and must also be in M'. The alternatives a_1, a_2, and a_3 are contained in every minimal upward covering set. But then M' is not minimal because the upward covering set M, which corresponds to the satisfying assignment stated above, is a strict subset of M'. Hence, e_1 cannot be contained in any minimal upward covering set.

To show the correctness of the construction in general, a key observation is stated in the following claim.

Claim 4.2 *Fix any j, $1 \leq j \leq \ell$. For each minimal upward covering set M for A, if M contains the alternative e_j then all other alternatives are contained in M as well (i.e., $A = M$).*

Proof of Claim 4.2. To simplify notation, we will prove the claim only for the case of $j = 1$. However, since there is nothing special about e_1 in our argument, the same property can be shown by an analogous argument for each j, $1 \leq j \leq \ell$.

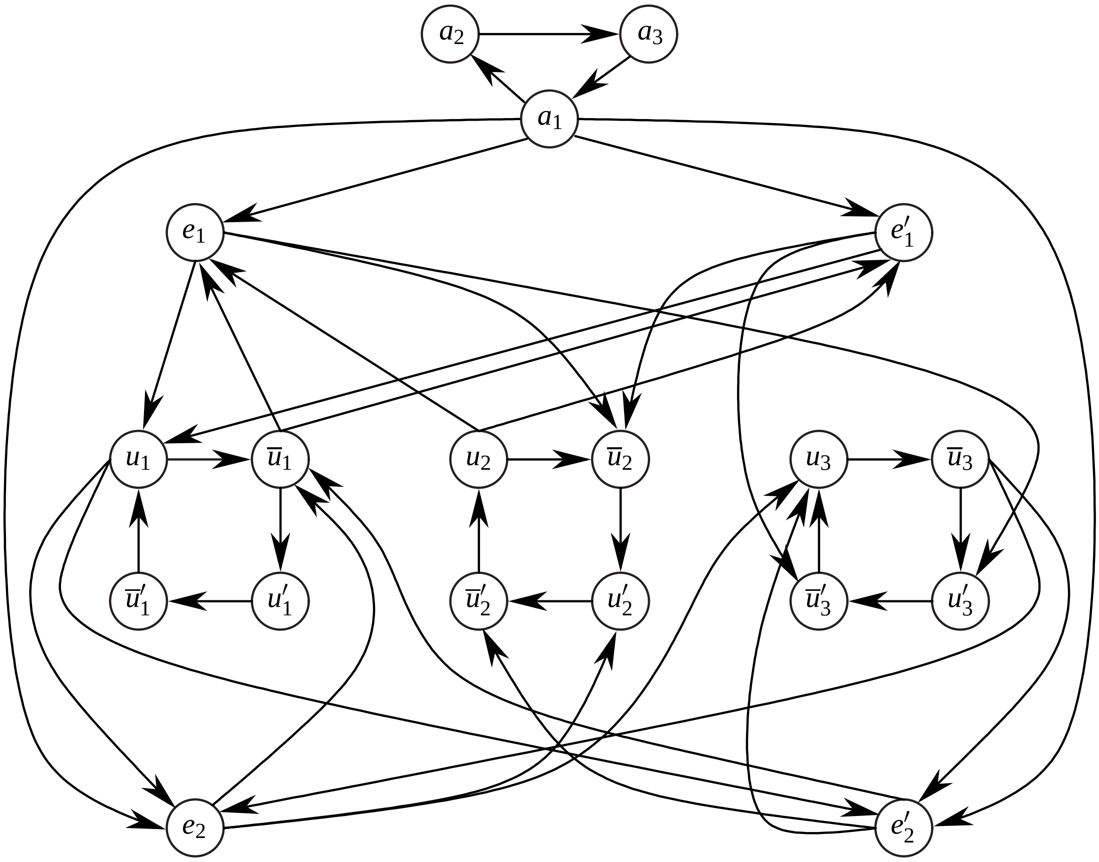


Figure 3: Dominance graph resulting from the formula $(\neg w_1 \vee w_2) \wedge (w_1 \vee \neg w_3)$ for Lemma 4.1.

Let M be any minimal upward covering set for A, and suppose that $e_1 \in M$. First note that the dominators of e_1 and e'_1 are always the same (albeit e_1 and e'_1 may dominate different alternatives). Thus, for each minimal upward covering set, either both e_1 and e'_1 are contained in it, or they both are not. Thus, since $e_1 \in M$, we have $e'_1 \in M$ as well.

Since the alternatives a_1, a_2, and a_3 form an undominated three-cycle, they each are contained in every minimal upward covering set for A. In particular, $\{a_1, a_2, a_3\} \subseteq M$. Furthermore, no alternative e_j or e'_j, $1 \leq j \leq \ell$, can upward cover any other alternative in M, because $a_1 \in M$ and a_1 dominates e_j and e'_j but none of the alternatives that are dominated by either e_j or e'_j. In particular, no alternative in any of the k four-cycles $u_i \succ \overline{u}_i \succ u'_i \succ \overline{u}'_i \succ u_i$ can be upward covered by any alternative e_j or e'_j, and so they each must be upward covered within their cycle. For each of these cycles, every minimal upward covering set for A must contain at least one of the sets $\{u_i, u'_i\}$ and $\{\overline{u}_i, \overline{u}'_i\}$, since at least one is needed to upward cover the other one.[2]

Since $e_1 \in M$ and by internal stability, we have that no alternative from M upward covers e_1. In addition to a_1, the alternatives dominating e_1 are u_i (for each i such that w_i occurs as a positive literal in f_1) and $\overline{u}_i$ (for

[2]The argument is analogous to that for the construction of Brandt and Fischer [BF08] in their proof of Theorem 3.1. However, in contrast with their construction, which implies that *either* $\{x_i, x'_i\}$ *or* $\{\overline{x}_i, \overline{x}'_i\}$, $1 \leq i \leq n$, *but not both*, must be contained in any minimal upward covering set for A (see Figure 1), our construction also allows for both $\{u_i, u'_i\}$ and $\{\overline{u}_i, \overline{u}'_i\}$ being contained in some minimal upward covering set for A. Informally stated, the reason is that, unlike the four-cycles in Figure 1, our four-cycles $u_i \succ \overline{u}_i \succ u'_i \succ \overline{u}'_i \succ u_i$ also have incoming edges.

each i such that w_i occurs as a negative literal in f_1).

First assume that, for some i, w_i occurs as a positive literal in f_1. Suppose that $\{u_i, u'_i\} \subseteq M$. If $\overline{u}'_i \notin M$ then e_1 would be upward covered by u_i, which is impossible. Thus, $\overline{u}'_i \in M$. But then $\overline{u}_i \in M$ as well, since u_i, the only alternative that could upward cover $\overline{u}_i$, is itself dominated by $\overline{u}'_i$. For the latter argument, recall that $\overline{u}_i$ cannot be upward covered by any e_j or e'_j. Thus, we have shown that $\{u_i, u'_i\} \subseteq M$ implies $\{\overline{u}_i, \overline{u}'_i\} \subseteq M$. Conversely, suppose that $\{\overline{u}_i, \overline{u}'_i\} \subseteq M$. Then u'_i is no longer upward covered by $\overline{u}_i$ and hence must be in M as well. The same holds for the alternative u_i, so $\{\overline{u}_i, \overline{u}'_i\} \subseteq M$ implies $\{u_i, u'_i\} \subseteq M$. Summing up, if $e_1 \in M$ then $\{u_i, u'_i, \overline{u}_i, \overline{u}'_i\} \subseteq M$ for each i such that w_i occurs as a positive literal in f_1.

By symmetry of the construction, an analogous argument shows that if $e_1 \in M$ then $\{u_i, u'_i, \overline{u}_i, \overline{u}'_i\} \subseteq M$ for each i such that w_i occurs as a negative literal in f_1.

Now, consider any i such that w_i does not occur in f_1. We have $e_1 \succ u'_i$ and $e'_1 \succ \overline{u}'_i$. Again, none of the sets $\{u_i, u'_i\}$ and $\{\overline{u}_i, \overline{u}'_i\}$ alone can be contained in M, since otherwise either u_i or $\overline{u}'_i$ would remain upward uncovered. Thus, $e_1 \in M$ again implies that $\{u_i, u'_i, \overline{u}_i, \overline{u}'_i\} \subseteq M$.

Now it is easy to see that, since $\bigcup_{1 \le i \le k}\{u_i, u'_i, \overline{u}_i, \overline{u}'_i\} \subseteq M$ and since a_1 cannot upward cover any of the e_j and e'_j, $1 \le j \le \ell$, external stability of M enforces that $\bigcup_{1<j \le \ell}\{e_j, e'_j\} \subseteq M$. Summing up, we have shown that if e_1 is contained in any minimal upward covering set M, then $M = A$. ❑ Claim 4.2

It remains to show that the given boolean formula φ is satisfiable if and only if e_1 is not contained in any minimal upward covering set for A.

From left to right, suppose there is a satisfying assignment $\alpha : W \to \{0,1\}$ for φ. Define the set

$$B = \{a_1, a_2, a_3\} \cup \{u_i, u'_i \mid \alpha(w_i) = 1\} \cup \{\overline{u}_i, \overline{u}'_i \mid \alpha(w_i) = 0\}.$$

Since every upward covering set for A must contain $\{a_1, a_2, a_3\}$ and at least one of the sets $\{u_i, u'_i\}$ and $\{\overline{u}_i, \overline{u}'_i\}$ for each i, $1 \le i \le k$, B is a (minimal) upward covering set for A. Let M be an arbitrary minimal upward covering set for A. By Claim 4.2, if e_1 were contained in M then we would have $M = A$. But since $B \subset A = M$, this contradicts the minimality of M. Thus $e_1 \notin M$.

From right to left, let M be an arbitrary minimal upward covering set for A and suppose $e_1 \notin M$. By Claim 4.2, if any of the e_j, $1 < j \le \ell$, were contained in M, it would follow that $e_1 \in M$, a contradiction. Thus, $\{e_j \mid 1 \le j \le \ell\} \cap M = \emptyset$. It follows that each e_j must be upward covered by some alternative in M. It is easy to see that for each j, $1 \le j \le \ell$, and for each i, $1 \le i \le k$, e_j is upward covered in $M \cup \{e_j\} \supseteq \{u_i, u'_i\}$ if w_i occurs in f_j as a positive literal, and e_j is upward covered in $M \cup \{e_j\} \supseteq \{\overline{u}_i, \overline{u}'_i\}$ if w_i occurs in e_j as a negative literal. It can never be the case that all four alternatives, $\{u_i, u'_i, \overline{u}_i, \overline{u}'_i\}$, are contained in M, because then either e_j would no longer be upward covered or the resulting set M was not minimal. Now, M induces a satisfying assignment for φ by setting, for each i, $1 \le i \le k$, $\alpha(w_i) = 1$ if $u_i \in M$, and $\alpha(w_i) = 0$ if $\overline{u}_i \in M$.

This concludes the proof that $\mathrm{MC_u}$-MEMBER is coNP-hard. ❑

The previous construction allows us to prove the hardness of several computational problems related to minimal upward covering sets.

Corollary 4.3 *Given a dominance graph $(A, \succ)$, it is* coNP-*hard to decide*

- *whether a given alternative is contained in* all *minimal upward covering sets for A,*
- *whether a given subset of A is a minimal upward covering set for A, and*
- *whether there is a unique minimal upward covering set for A.*

Proof. It follows from Claim 4.2 that φ is not satisfiable if and only if the entire set of alternatives A is a (unique) minimal upward covering set for A. Furthermore, if φ is satisfiable, there exists more than one minimal upward covering set for A and none of them contains e_1 (provided that φ has more than one satisfying assignments, which can be ensured by adding a dummy variable such that the satisfiability of the formula is not affected). The first and the second problem are also *contained* in coNP, because they can be decided in the positive by checking whether there does *not* exist an upward covering set that satisfies certain properties. Thus, the first and the second problem are coNP-complete. ❑

The first statement of the corollary above was already shown by Brandt and Fischer [BF08]. However, their proof—which uses essentially the reduction from the proof of Theorem 3.1, except that they start from the coNP-complete problem VALIDITY (which asks whether a given formula is valid, i.e., whether it is true under every assignment [Pap94])—does not yield any of the other coNP-hardness results, including coNP-hardness of $\mathrm{MC_u}$-MEMBER.

An important consequence of the proof of Corollary 4.3 is the following.

Corollary 4.4 *Minimal upward covering sets cannot be found in polynomial time unless* $\mathrm{P} = \mathrm{NP}$.

Proof. Consider the problem of deciding whether there exists a nontrivial minimal upward covering set, i.e., a minimal upward covering set that does *not* contain all alternatives. By the construction from the proof of Lemma 4.1 that is applied in proving Corollary 4.3, there exists a trivial minimal upward covering set for A (i.e., a minimal upward covering set containing all alternatives in A) if and only if this set is the only minimal upward covering set for A. Thus, the coNP-hardness proof for the problem of deciding whether there is a unique minimal upward covering set for A (see the proof of Corollary 4.3) immediately implies that the problem of deciding whether there is a nontrivial minimal upward covering set for A is NP-hard. However, since the latter problem can easily be reduced to the search problem (because the search problem, when used as a function oracle, will yield the set of all alternatives if and only if this set is the only minimal upward covering set for A), it follows that the search problem cannot be computed in polynomial time unless $\mathrm{P} = \mathrm{NP}$. ❑

Now that we have established that $\mathrm{MC_u}$-MEMBER is NP-hard as well as coNP-hard, we will raise the lower bound to Θ_2^p-hardness. Wagner provided a sufficient condition for proving Θ_2^p-hardness that was useful in various other contexts (see, e.g., [Wag87, HHR97, HR98, HW02, HRS06]) and is stated here as Lemma 4.5.

Lemma 4.5 ([Wag87]) *Let S be some* NP*-complete problem, and let T be any set. If there exists a polynomial-time computable function f such that, for all $m \geq 1$ and all strings $x_1, x_2, \ldots, x_{2m}$ satisfying that if $x_j \in S$ then $x_{j-1} \in S$, $1 < j \leq 2m$, we have*

$$\|\{i \mid x_i \in S\}\| \text{ is odd} \iff f(x_1, x_2, \ldots, x_{2m}) \in T, \tag{4.1}$$

then T is Θ_2^p-hard.

We will apply Lemma 4.5 as well. In contrast with those previous results, however, one subtlety in our construction is due to the fact that minimality in $\mathrm{MC_u}$-MEMBER is defined in terms of set inclusion, not in terms of the cardinality of minimal upward covering sets. For example, recall Wagner's Θ_2^p-completeness result for testing whether the size of a maximum clique in a given graph is an odd number [Wag87]. One key ingredient in his proof is to define an associative operation on graphs, $\bowtie$, such that for any two graphs G and H, the size of a maximum clique in $G \bowtie H$ equals the sum of the sizes of a maximum clique in G and one in H. This operation is quite simple: Just connect every vertex of G with every vertex of H. In contrast, since minimality for minimal upward covering sets is defined in terms of set inclusion, it is not at all obvious how to define a similarly simple operation on dominance graphs such that the minimal upward covering sets in the given graphs are related to the minimal upward covering sets in the connected graph in a similarly useful way. Nonetheless, we will prove that $\mathrm{MC_u}$-MEMBER is Θ_2^p-hard by applying Lemma 4.5 and making use of the constructions presented in the proofs of Theorem 3.1 and Lemma 4.1.

We apply Wagner's Lemma with the NP-complete problem $S = \mathrm{SAT}$ and with $T = \mathrm{MC_u}$-MEMBER. Fix an arbitrary $m \geq 1$ and let $\varphi_1, \varphi_2, \ldots, \varphi_{2m}$ be $2m$ boolean formulas such that if φ_j is satisfiable then so is φ_{j-1}, for each j, $1 < j \leq 2m$. Without loss of generality, we assume that for each j, $1 \leq j \leq 2m$, the first variable of φ_j does not occur in all clauses of φ_j. It is easy to see that if φ_j does not have this property, it can be transformed into a formula that does have it, without affecting the satisfiability of the formula.

We will now define a polynomial-time computable function f, which maps the given $2m$ boolean formulas to an instance of $\mathrm{MC_u}$-MEMBER such that (4.1) is satisfied. First, to construct a dominance graph $(A, \succ)$ as part of this instance, define $A = \bigcup_{j=1}^{2m} A_j$ and the dominance relation $\succ$ on A by

$$\left(\bigcup_{j=1}^{2m} \succ_j\right) \cup \left(\bigcup_{i=1}^{m} \{(u'_{1,2i}, d_{2i-1}), (\overline{u}'_{1,2i}, d_{2i-1})\}\right) \cup \left(\bigcup_{i=2}^{m} \{(d_{2i-1}, z) \mid z \in A_{2i-2}\}\right),$$

where we use the following notation:

1. For each i, $1 \leq i \leq m$, let $(A_{2i-1}, \succ_{2i-1})$ be the dominance graph that results from the formula φ_{2i-1} according to Brandt and Fischer's construction given in the proof sketch of Theorem 3.1. We use the same names for the alternatives in A_{2i-1} as in that proof sketch, except that we attach the subscript $2i-1$. For example, alternative d from the proof sketch of Theorem 3.1 now becomes d_{2i-1}, x_1 becomes $x_{1,2i-1}$, y_1 becomes $y_{1,2i-1}$, and so on.

2. For each i, $1 \leq i \leq m$, let $(A_{2i}, \succ_{2i})$ be the dominance graph that results from the formula φ_{2i} according to the construction given in the proof of Lemma 4.1. We use the same names for the alternatives in A_{2i} as in that proof, except that we attach the subscript $2i$. For example, alternative a_1 from the proof of Lemma 4.1 now becomes $a_{1,2i}$, e_1 becomes $e_{1,2i}$, u_1 becomes $u_{1,2i}$, and so on.

3. For each i, $1 \leq i \leq m$, connect the dominance graphs $(A_{2i-1}, \succ_{2i-1})$ and $(A_{2i}, \succ_{2i})$ as follows. Let $u_{1,2i}, \overline{u}_{1,2i}, u'_{1,2i}, \overline{u}'_{1,2i} \in A_{2i}$ be the four alternatives in the cycle corresponding to the first variable of φ_{2i}. Then both $u'_{1,2i}$ and $\overline{u}'_{1,2i}$ dominate d_{2i-1}. The resulting dominance graph is denoted by $(B_i, \succ_i^B)$.

4. Connect the m dominance graphs $(B_i, \succ_i^B)$, $1 \leq i \leq m$, as follows: For each i, $2 \leq i \leq m$, d_{2i-1} dominates all alternatives in A_{2i-2}.

The dominance graph $(A, \succ)$ is sketched in Figure 4. Now, letting d_1 be the designated element of A whose membership in some minimal upward covering set for A with respect to $\succ$ is in question, we obtain an instance of $\mathrm{MC_u}$-MEMBER, $(A, \succ, d_1)$. Clearly, the function f defined by $f(\varphi_1, \varphi_2, \ldots, \varphi_{2m}) = (A, \succ, d_1)$ is computable in polynomial time.

Before we show—via Lemma 4.5 and the reduction f defined above—that $\mathrm{MC_u}$-MEMBER is Θ_2^p-hard, let us first consider the dominance graph $(B_i, \succ_i^B)$ separately,[3] for any fixed i with $1 \leq i \leq m$. Doing so will simplify

[3]Note that our argument about $(B_i, \succ_i^B)$ shows, in effect, DP-hardness of $\mathrm{MC_u}$-MEMBER, where DP is the class of differences of any two NP sets [PY84]. Note that DP is the second level of the boolean hierarchy over NP (see Cai et al. [CGH+88, CGH+89]), and it holds that $\mathrm{NP} \cup \mathrm{coNP} \subseteq \mathrm{DP} \subseteq \Theta_2^p$. Wagner [Wag87] proved appropriate analogs of Lemma 4.5 for each level of the boolean hierarchy. In particalar, the analogous criterion for DP-hardness is obtained by using the wording of Lemma 4.5 except with the value of $m = 1$ being fixed.

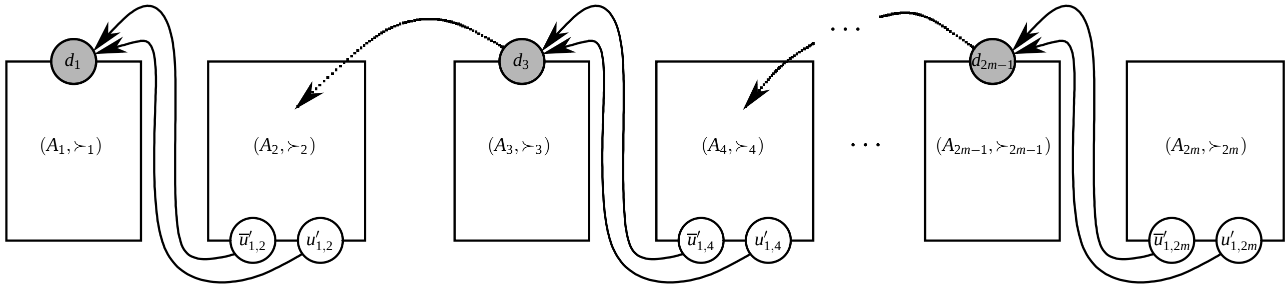


Figure 4: Dominance graph $(A,\succ)$ from the proof of Theorem 3.2. Most alternatives and all arrows within A_j, $1 \le j \le 2m$, are omitted; only arrows between alternatives in distinct A_i and A_j, $i \neq j$, are shown. A dotted arrow from d_{2i-1}, $2 \le i \le m$, indicates that d_{2i-1} dominates all alternatives in A_{2i-2}.

our argument for the whole dominance graph $(A,\succ)$. Recall that $(B_i,\succ_i^B)$ results from the formulas φ_{2i-1} and φ_{2i} via the proofs of Theorem 3.1 and Lemma 4.1, respectively. Whether or not alternative d_{2i-1} is contained in some minimal upward covering set for $(B_i,\succ_i^B)$ depends on the satisfiability of φ_{2i-1} and φ_{2i}. Accordingly, we below distinguish three cases. Note that, by our assumption on how the formulas are ordered, the fourth case (i.e., $\varphi_{2i-1} \notin \mathrm{SAT}$ and $\varphi_{2i} \in \mathrm{SAT}$) cannot occur.

Case 1: $\varphi_{2i-1} \in \mathrm{SAT}$ and $\varphi_{2i} \in \mathrm{SAT}$. Since φ_{2i} is satisfiable, it follows from the proof of Lemma 4.1 that for each minimal upward covering set M for $(B_i,\succ_i^B)$, either $\{u_{1,2i}, u'_{1,2i}\} \subseteq M$ or $\{\overline{u}_{1,2i}, \overline{u}'_{1,2i}\} \subseteq M$, but not both, and that none of the $e_{j,2i}$ and $e'_{j,2i}$ is in M. If $\overline{u}'_{1,2i} \in M$ but $u'_{1,2i} \notin M$, then $d_{2i-1} \notin \mathrm{UC}_u(M)$, since $\overline{u}'_{1,2i}$ upward covers d_{2i-1} within M. If $u'_{1,2i} \in M$ but $\overline{u}_{1,2i} \notin M$, then $d_{2i-1} \notin \mathrm{UC}_u(M)$, since $u'_{1,2i}$ upward covers d_{2i-1} within M. Hence, by internal stability, d_{2i-1} is not contained in M.

Case 2: $\varphi_{2i-1} \notin \mathrm{SAT}$ and $\varphi_{2i} \notin \mathrm{SAT}$. Since $\varphi_{2i-1} \notin \mathrm{SAT}$, it follows from the proof of Theorem 3.1 that each minimal upward covering set M for $(B_i,\succ_i^B)$ contains at least one alternative $y_{j,2i-1}$ (corresponding to some clause of φ_{2i-1}) that upward covers d_{2i-1}. Thus d_{2i-1} cannot be in M, again by internal stability.

Case 3: $\varphi_{2i-1} \in \mathrm{SAT}$ and $\varphi_{2i} \notin \mathrm{SAT}$. Since $\varphi_{2i-1} \in \mathrm{SAT}$, it follows from the proof of Theorem 3.1 that there exists a minimal upward covering set M' for $(A_{2i-1},\succ_{2i-1})$ that corresponds to a satisfying truth assignment for φ_{2i-1}. In particular, none of the $y_{j,2i-1}$ is in M'. On the other hand, since $\varphi_{2i} \notin \mathrm{SAT}$, it follows from the proof of Lemma 4.1 that A_{2i} is the only minimal upward covering set for $(A_{2i},\succ_{2i})$. Define $M = M' \cup A_{2i}$. It is easy to see that M is a minimal upward covering set for $(B_i,\succ_i^B)$, since the only edges between A_{2i-1} and A_{2i} are those from $\overline{u}'_{1,2i}$ and $u'_{1,2i}$ to d_{2i-1}, and both $\overline{u}'_{1,2i}$ and $u'_{1,2i}$ are dominated by elements in M not dominating d_{2i-1}.

We now show that $d_{2i-1} \in M$. Note that $\overline{u}'_{1,2i}$, $u'_{1,2i}$, and the $y_{j,2i-1}$ are the only alternatives in B_i that dominate d_{2i-1}. Since none of the $y_{j,2i-1}$ is in M, they do not upward cover d_{2i-1}. Also, $u'_{1,2i}$ doesn't upward cover d_{2i-1}, since $\overline{u}_{1,2i} \in M$ and $\overline{u}_{1,2i}$ dominates $u'_{1,2i}$ but not d_{2i-1}. On the other hand, by our assumption that the first variable of φ_{2i} does not occur in all clauses, there exist alternatives $e_{j,2i}$ and $e'_{j,2i}$ in M that dominate $\overline{u}'_{1,2i}$ but not d_{2i-1}, so $\overline{u}'_{1,2i}$ doesn't upward cover d_{2i-1} either. Thus $d_{2i-1} \in M$.

For each i, $1 \le i \le m$, let M_i be the minimal upward covering set for $(B_i,\succ_i^B)$ according to the above cases. Note that the minimal upward covering set M_m for $(B_m,\succ_m^B)$ must be contained in every minimal upward covering set for $(A, \succ)$, since no alternative in $A - B_m$ dominates any alternative in B_m. On the other hand, for each i, $1 \leq i < m$, no alternative in B_i can be upward covered by d_{2i+1} (which is the only element in $A - B_i$ that dominates any of the elements of B_i), since d_{2i+1} is dominated within every minimal upward covering set for B_{i+1} (and, in particular, within M_{i+1}). Thus, each of the sets M_i, $1 \leq i \leq m$, must be contained in every minimal upward covering set for $(A, \succ)$.

Now, to apply Lemma 4.5, we prove (4.1), which here amounts to showing the following equivalence:

$$\|\{i \mid \varphi_i \in \mathrm{SAT}\}\| \text{ is odd} \iff d_1 \text{ is contained in some minimal upward covering set } M \text{ for } A. \quad (4.2)$$

To show (4.2) from left to right, suppose $\|\{i \mid \varphi_i \in \mathrm{SAT}\}\|$ is odd. Recall that for each j, $1 < j \leq 2m$, if φ_j is satisfiable then so is φ_{j-1}. Thus, there exists some i, $1 \leq i \leq m$, such that $\varphi_1, \dots, \varphi_{2i-1} \in \mathrm{SAT}$ and $\varphi_{2i}, \dots, \varphi_{2m} \notin \mathrm{SAT}$. In Case 3 of the case distinction above we have seen that there is some minimal upward covering set for $(B_i, \succ_i^B)$—call it M_i—that corresponds to a satisfying assignment of φ_{2i-1} and that contains all alternatives of A_{2i}. In particular, M_i contains d_{2i-1}. For each $j \neq i$, $1 \leq j \leq m$, let M_j be some minimal upward covering set for $(B_j, \succ_j^B)$ according to Case 1 (if $j < i$) and Case 2 (if $j > i$).

In Case 1 we have seen that d_{2i-3} is upward covered either by $\overline{u}'_{1,2i-3}$ or by $u'_{1,2i-3}$. This is no longer the case, since d_{2i-1} is in M_i and it dominates all alternatives in A_{2i-2} but not d_{2i-3}. By assumption, φ_{2i-3} is satisfiable, so there exists a minimal upward covering set, which contains d_{2i-3} as well. Thus, setting

$$M = \{d_1, d_3, \dots, d_{2i-1}\} \cup \bigcup_{1 \leq j \leq m} M_j,$$

it follows that M is a minimal upward covering set for $(A, \succ)$ containing d_1.

To show (4.2) from right to left, suppose that $\|\{i \mid \varphi_i \in \mathrm{SAT}\}\|$ is even. For a contradiction, suppose that there exists some minimal upward covering set M for $(A, \succ)$ that contains d_1. If $\varphi_1 \notin \mathrm{SAT}$ then we immediately obtain a contradiction by the argument in the proof of Theorem 3.1. On the other hand, if $\varphi_1 \in \mathrm{SAT}$ then our assumption that $\|\{i \mid \varphi_i \in \mathrm{SAT}\}\|$ is even implies that $\varphi_2 \in \mathrm{SAT}$. It follows from the proof of Lemma 4.1 that every minimal upward covering set for $(A, \succ)$ (thus, in particular, M) contains either $\{u_{1,2i}, u'_{1,2i}\}$ or $\{\overline{u}_{1,2i}, \overline{u}'_{1,2i}\}$, but not both, and that none of the $e_{j,2i}$ and $e'_{j,2i}$ is in M. By the argument presented in Case 3 above, the only way to prevent d_1 from being upward covered by an element of M, either $u'_{1,2}$ or $\overline{u}'_{1,2}$, is to include d_3 in M as well.[4] By applying the same argument $m-1$ times, we will eventually reach a contradiction, since $d_{2m-1} \in M$ can no longer be prevented from being upward covered by an element of M, either $u'_{1,2m}$ or $\overline{u}'_{1,2m}$. Thus, no minimal upward covering set M for $(A, \succ)$ contains d_1, which completes the proof of (4.2).

Since (4.2) is true, Lemma 4.5 implies that $\mathrm{MC_u}$-MEMBER is Θ_2^p-hard.

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[4]This implies that d_1 is not upward covered by either $u'_{1,2}$ or $\overline{u}'_{1,2}$, since d_3 dominates them both but not d_1.

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