On the Complexity of Iterated Weak Dominance in Constant-Sum Games

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Abstract. In game theory, a player's action is said to be weakly dominated if there exists another action that, with respect to what the other players do, is never worse and sometimes strictly better. We investigate the computational complexity of the process of iteratively eliminating weakly dominated actions (IWD) in two-player constant-sum games, i.e., games in which the interests of both players are diametrically opposed. It turns out that deciding whether an action is eliminable via IWD is feasible in polynomial time whereas deciding whether a given subgame is reachable via IWD is NP-complete. The latter result is quite surprising as we are not aware of other natural computational problems that are intractable in constant-sum games. Furthermore, we slightly improve a result by Conitzer and Sandholm [6] by showing that typical problems associated with IWD in win-lose games with at most one winner are NP-complete.

1 Introduction

A simple and indisputable conviction in game theory is that a player need not bother to consider an action that yields less payoff than some other action *no matter* what all the other players do (see, e.g., [11]). In game-theoretic terms, such an action is *strictly dominated*. Similarly, one says that an action is *weakly dominated* if there exists another action that, with respect to what the other players do, is never worse and sometimes strictly better. An action that is not weakly dominated is also said to be *admissible*. When a (strictly or weakly) dominated action is eliminated from a player's consideration, it may be possible that a previously undominated action of another player becomes dominated. Thus, based on the mutual rational belief that (some) dominated actions will not be played, one can define an iterative process of eliminating actions. It is well-known that this process invariably leads to the same subgame no matter in which order strictly dominated actions are eliminated whereas this is not the case for weak dominance (see, e.g., [1, 19]). The dependence on the order of elimination gives rise to some combinatorial difficulties as witnessed by the NP-completeness of various computational problems related to iterated weak dominance [8, 6]. By contrast, the corresponding problems for iterated strict dominance are computationally tractable. This disparity has also become apparent in the complexity analysis of other solution concepts based on dominance [4].

We investigate the computational complexity of iterated weak dominance (IWD)—or *iterated admissibility*—in *two-player constant-sum* games, i.e., games in which the interests of both players are diametrically opposed. Our analysis is restricted to dominance by pure strategies although most of our results can be readily applied to mixed strategies as well (see Section 6). It turns out that deciding whether an action is eliminable via IWD is feasible in polynomial time whereas deciding whether a given subgame is reachable via IWD is NP-complete. The latter result is quite surprising as we are not aware of other natural computational problems that are intractable in constant-sum games. Furthermore, we slightly improve a result by Conitzer and Sandholm [6] by showing that typical problems associated with IWD in win-lose games *with at most one winner* are NP-complete.

Iterated weak and strict dominance are well-established solution concepts, which have a long history and occur in virtually every textbook on game theory. The publication of Bernheim [2] and Pearce [16] instigated a renewed discussion concerning their formal and intuitive connections with rationalizability and the epistemic foundations of solution concepts [3, 17], the stability of equilibria [10], and backward induction solutions [7, 18]. It cannot be said that iterated weak dominance has left the arena entirely unscathed. Unlike iterated strict dominance, proper epistemic foundations for iterated weak dominance are pretty hard to come by. In particular, Samuelson [17] showed that common knowledge of admissibility does not imply iterated weak dominance. Nevertheless, IWD has been argued to have its place as a tool in the analysis of games (see, e.g., [14, 13], for discussions). Our aim, however, is by no means to pass judgement on iterated weak dominance as a solution concept as such. Rather, our focus is on the computational aspects of IWD in two-player zero-sum and win-lose games with at most one winner. As mentioned above, the fact that some of these problems turn out to be NP-hard is interesting and surprising in its own right.

After having introduced our formal framework (Section 2), we introduce the auxiliary concept of a regionalized game in Section 3. We prove that regionalized games may be used as a convenient tool in the proofs of our hardness results. In Section 4 we deal with the computational complexity of the reachability and eliminability problems in two-player constant-sum games. Finally, we address the same problems for win-lose games that allow at most one winner in Section 5. The proofs of some of our results are deferred to the appendix.

2 Preliminaries

A *two-player game* $\Gamma = (A_1, A_2, u)$ is given by a finite set A_1 of actions of player 1, a finite set A_2 of actions of player 2, and a utility function $u : A_1 \times A_2 \to \mathbb{R} \times \mathbb{R}$. We also have A denote $A_1 \cup A_2$ and write $u_1(a, b) = x$ and $u_2(a, b) = y$ if $u(a, b) = (x, y)$. Both players are assumed to choose one of their actions simultaneously. If player 1 chooses a and player 2 chooses b, their payoffs will be $u_1(a, b)$ and $u_2(a, b)$, respectively.

A two-player game is called *constant-sum* if $u_1(a,b) + u_2(a,b) = u_1(c,d) + u_2(c,d)$ for all $a, c \in A_1$ and $b, d \in A_2$. It is convenient to write down the payoffs of a game in a matrix with rows indexed by the actions of player 1 and columns indexed by the actions of players 2.

Consider a game and let $a, b \in A_1$ be two actions of player 1. Then a is said to *weakly dominate* b *at* $c \in A_2$ if $u_1(a,c) > u_1(b,c)$ and for all $d \in A_2$, $u_1(a,d) \geq u_1(b,d)$. More generally, a is said to weakly dominate b if a weakly dominates b at c for some $c \in A_2$. We further say that $c \in A_2$ *backs* the elimination of b by a if $u_1(a,c) > u_1(b,c)$, and *blocks* the elimination of b by a if $u_1(a,c) < u_1(b,c)$. Dominance, backing, and blocking for actions of player 2 is defined analogously. Obviously an action is dominated by another action of the same player if some action of the other player backs the elimination, and none of them blocks it. As the remainder of this paper only concerns (iterated) weak dominance, we will drop the qualification 'weak' and by 'dominance' understand weak dominance, unless stated otherwise.

An *elimination sequence* of a game $\Gamma = (A_1, A_2, u)$ is a finite sequence $\Sigma = (\Sigma_1, \ldots, \Sigma_k)$ of subsets of actions in $A = A_1 \cup A_2$. For a game $\Gamma = (A_1, A_2, u)$ and an elimination sequence $\Sigma = (\Sigma_1, \ldots, \Sigma_k)$ of Γ we have $\Gamma(\Sigma)$ denote the subgame where the actions in $\Sigma_1 \cup \cdots \cup \Sigma_k$ have been removed, i.e., $\Gamma(\Sigma) = (A'_1, A'_2, u')$ where $A'_1 = A_1 \setminus (\Sigma_1 \cup \cdots \cup \Sigma_k)$ and $A'_2 = A_2 \setminus (\Sigma_1 \cup \cdots \cup \Sigma_k)$ and u' is the restriction of u to $A'_1 \times A'_2$. The *validity* of elimination sequences is then defined inductively: the empty sequence ϵ is valid for every game and an elimination sequence $(\Sigma_1, \ldots, \Sigma_m, \Sigma_{m+1})$ is valid in Γ if $(\Sigma_1, \ldots, \Sigma_m)$ is valid in Γ and every action $a \in \Sigma_{m+1}$ is dominated in $\Gamma(\Sigma_1, \ldots, \Sigma_m)$. If in $(\Sigma_1, \ldots, \Sigma_m)$ for each $1 \leq i \leq m$, Σ_i is a singleton, we say the elimination sequence is *simple*. Simple elimination sequences we usually write as sequences $\sigma = (\sigma_1, \ldots, \sigma_m)$ of actions in A.

Let Γ be a game. Then, an action a is called *eliminable by* b *at* c in Γ if there exists a valid elimination sequence Σ such that a is dominated by b at c in $\Gamma(\Sigma)$. Action a is *eliminable* in Γ if there are actions b and c such that a is eliminable by b at c. A subgame Γ' of Γ is *reachable* from Γ if there exists a valid elimination sequence Σ such that $\Gamma(\Sigma) = \Gamma'$. Furthermore Γ is called *solvable* if some game $\Gamma' = (A'_1, A'_2, u')$ with $|A'_1| = |A'_2| = 1$ is reachable from Γ. Finally, we say Γ is *irreducible* if none of its actions is dominated.

We assume familiarity with the theory of complexity, in particular with the complexity classes P and NP and the canonical problem *3SAT* (see, e.g., [15]).

3 Regions and Regionalized Games

An essential building block of our hardness proofs are *regionalized games*.

Definition 1. *A* regionalized two-player game *is a tuple* (Γ, X_1, X_2) *consisting of a two-player game* $\Gamma = (A_1, A_2, u)$ *and partitions* X_1 *and* X_2 *of* A_1 *and* A_2*, respectively. The elements of* X_1 *and* X_2 *are also called* regions*.*

For regionalized games the concept of a valid elimination sequence is modified, so as to allow only eliminations of actions that are dominated by other actions in the same region.

Definition 2. *A* valid elimination sequence *for a regionalized game* (Γ, X_1, X_2) *is a sequence* $\Sigma = (\Sigma_1, \dots, \Sigma_k)$ *for* Γ *such that for each* i *with* $1 \leq i \leq k$ *and each* $a \in \Sigma_i$*, there is some action* b *and some* $x \in X_1 \cup X_2$ *such that* $a, b \in x$ *and* b *dominates* a *in* $\Gamma(\Sigma_1, \dots, \Sigma_{i-1})$*.*

The following lemma shows that any regionalized two-player game can be transformed in polynomial time into a non-regionalized two-player game with the same valid elimination sequences. The significance of this result is that for the computational problems we consider—reachability of (irreducible) subgames, eliminability and solvability—we can restrict ourselves to regionalized games, which are often more practical for and afford more insight into the constructions used in our hardness proofs than games without regions.

Lemma 1. *For each regionalized game* (Γ, X_1, X_2) *with* $\Gamma = (A_1, A_2, u)$*, there is a game* $\Gamma' = (A'_1, A'_2, u')$ *computable in polynomial time such that the valid elimination sequences of* Γ' *and* (Γ, X_1, X_2) *coincide:*

$$\{\Sigma \colon \Sigma \text{ a valid sequence in } \Gamma'\} = \{\Sigma \colon \Sigma \text{ a valid sequence in } (\Gamma, X_1, X_2)\}.$$

Moreover, $u'(a, b) \in \{(0, 1), (1, 0)\}$ *for all* $a \in A'_1 \setminus A_1$ *and* $b \in A'_2 \setminus A_2$*.*

4 Two-Player Constant-sum Games

We show that subgame reachability is NP-complete even in games that only allow the outcomes $(0, 1)$ and $(1, 0)$. This may be attributed to the order dependence of IWD. In Section 4.2 we find that for two-player constant-sum games a weak form of order independence can be salvaged, which allows us to formulate an efficient algorithm for the eliminability problem. We first show that in the case of two-player zero-sum games we can restrict our attention to *simple* elimination sequences.

Lemma 2. *Let* Γ *be a two-player constant-sum game and* $\Sigma = (\Sigma_1, \dots, \Sigma_m)$ *a valid elimination sequence. Then, there is a simple elimination sequence* $\sigma = (\sigma_1, \dots, \sigma_k)$ *with* $\{\sigma_1, \dots, \sigma_m\} = \Sigma_1 \cup \dots \cup \Sigma_m$ *that is also valid in* Γ*.*

Since every simple elimination sequence is an elimination sequence, it follows that a subgame of a two-player zero-sum game is reachable if and only if it is reachable by a simple elimination sequence. Analogous statements hold for eliminability and solvability in two-player zero-sum games.

Lemma 2 does not hold for general strategic games. In particular it fails for games with outcomes in $\{(0, 0), (0, 1)(1, 0)\}$, as the game in Figure 1 shows.

	y	y'
x	$(0,0)$	$(0,1)$
x'	$(1,0)$	$(0,0)$

Fig. 1. Both x and y are weakly dominated in the game above. Hence, the elimination sequence $\{x, y\}$ is valid. However, neither of the simple elimination sequences (x, y) and (y, x) is valid.

4.1 Reachability

We first show that subgame reachability in constant-sum games is intractable.

Theorem 1. *Given constant-sum games Γ and Γ', deciding whether Γ' is reachable from Γ is NP-complete, even when restricted to outcomes $(0,1)$ and $(1,0)$ and Γ' is to be irreducible.*

Proof. For membership in NP consider arbitrary constant-sum games Γ and Γ'. Given an elimination sequence $\sigma = (\sigma_1, \ldots, \sigma_k)$, it can clearly be decided in polynomial time whether σ is a valid elimination sequence for (Γ, X_1, X_2) such that $\Gamma' = \Gamma(\sigma)$.

The proof of hardness proceeds by a reduction from *3SAT*. By virtue of Lemma 1 it suffices to prove this for regionalized games. Consider an arbitrary *3CNF*$\varphi = C_1 \wedge \cdots \wedge C_k$, where each $C_i = (\lambda_i^1 \vee \lambda_i^2 \vee \lambda_i^3)$ is a clause and each λ_i^j is a literal, for $1 \leq i \leq k$ and $1 \leq j \leq 3$. Define the regionalized game $(\Gamma_\varphi, X_1, X_2)$, with $\Gamma_\varphi = (A_1, A_2, u)$ as follows.

$$
\begin{aligned}
A_1 &= \{p, \neg p, \neg(p \wedge \neg p) \colon p \text{ a variable in } \varphi\} \\
&\quad \cup \{C_i, (\lambda_i^1, i), (\lambda_i^2, i), (\lambda_i^3, i) \colon C_i \text{ a clause in } \varphi\} \\
&\quad \cup \{e\} \\
A_2 &= \{p, \neg p \colon p \text{ a variable in } \varphi\} \cup \{a, b\} \\
X_1 &= \{\{p, \neg p, \neg(p \wedge \neg p)\} \colon p \text{ a variable in } \varphi\} \\
&\quad \cup \{\{C_i, (\lambda_i^1, i), (\lambda_i^2, i), (\lambda_i^3, i)\} \colon C_i \text{ a clause in } \varphi\} \\
&\quad \cup \{\{e\}\} \\
X_2 &= \{\{p, \neg p \colon p \text{ a variable in } \varphi\} \cup \{a, b\}\} = \{A_2\}
\end{aligned}
$$

For each propositional variable p occurring in φ, the payoffs in rows p, $\neg p$ and $\neg(p \wedge \neg p)$ are defined as in the following table, where q is a typical variable in φ distinct from p.

	p	$\neg p$		q	$\neg q$		a	b
p	$(1,0)$	$(0,1)$		$(0,1)$	$(0,1)$		$(1,0)$	$(0,1)$
$\neg p$	$(0,1)$	$(1,0)$		$(0,1)$	$(0,1)$		$(1,0)$	$(0,1)$
$\neg(p \wedge \neg p)$	$(1,0)$	$(1,0)$		$(0,1)$	$(0,1)$		$(0,1)$	$(0,1)$

Due to the regionalization, $\neg(p\wedge\neg p)$ can be eliminated only by row p or row $\neg p$. Moreover, column a is the only action backing the elimination of $\neg(p\wedge\neg p)$. Also, at least one of the columns p and $\neg p$ needs to be removed (by column b) before $\neg(p\wedge\neg p)$ can be eliminated. Intuitively, removing column p means setting variable p to false, removing column $\neg p$, setting variable p to true, thus choosing a valuation.

Also for each i with $1 \leq i \leq k$, the payoffs in rows C_i, (λ_i^1, i), (λ_i^2, i), (λ_i^3, i) depend on the literals occurring in C_i. In the table below, $\overline{\lambda}_i^j = \neg p$, if $\lambda_i^j = p$, and $\overline{\lambda}_i^j = p$, if $\lambda_i^j = \neg p$. Also, we assume $i \neq m$.

	λ_i^1	$\overline{\lambda}_i^1$	λ_i^2	$\overline{\lambda}_i^2$	λ_i^3	$\overline{\lambda}_i^3$		λ_m^j	$\overline{\lambda}_m^j$		a	b
(λ_i^1, i)	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$		$(0,1)$	$(0,1)$		$(0,1)$	$(0,1)$
(λ_i^2, i)	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$		$(0,1)$	$(0,1)$		$(0,1)$	$(0,1)$
(λ_i^3, i)	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$		$(0,1)$	$(0,1)$		$(0,1)$	$(0,1)$
C_i	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$		$(0,1)$	$(0,1)$		$(1,0)$	$(0,1)$

Thus, the only columns backing the elimination of C_i are λ_i^1, λ_i^2 and λ_i^3. Also notice that column a blocks the elimination of C_i. Nevertheless, as we saw above, column a is essential to the elimination of the rows $\neg(p\wedge\neg p)$. Intuitively, this signifies that a valuation needs to be chosen before any of the rows C_i is eliminated.

Finally, let $u(e, y) = (1, 0)$ if $y \neq b$, and $u(e, y) = (0, 1)$.

	λ_1^1	$\overline{\lambda}_1^1$	$\cdots$	λ_k^3	$\overline{\lambda}_k^3$	a	b
e	$(1,0)$	$(1,0)$	$\cdots$	$(1,0)$	$(1,0)$	$(1,0)$	$(0,1)$

Observe that row e is the only action in its region and as such cannot be eliminated. Also, e backs the elimination of every column by b.

Now define $(\Gamma'_\varphi, X'_1, X'_2)$ with $\Gamma'_\varphi = (A'_1, A'_2, u')$ such that

$$\begin{aligned} A'_1 &= \{p, \neg p\colon p \text{ a variable in } \varphi\} \\ A'_2 &= \{b\}. \end{aligned}$$

Moreover, we have u', X'_1 and X'_2 appropriately restricted to A'_1 and A'_2, i.e., $u' = u|_{A'_1 \times A'_2}$, $X'_1 = \{x \cap A'_1 \colon x \in X_1\}$ and $X'_2 = \{x \cap A'_2 \colon x \in X_2\}$. It is easily appreciated that in $(\Gamma'_\varphi, X'_1, X'_2)$ there are no actions that can be eliminated, i.e., $(\Gamma'_\varphi, X'_1, X'_2)$ is irreducible.

We now prove that φ is satisfiable if and only if $(\Gamma'_\varphi, X'_1, X'_2)$ is reachable from $(\Gamma_\varphi, X_1, X_2)$.

Assume that φ is satisfiable and let v be a valuation satisfying φ. Now let b eliminate all columns representing a literal λ that is set to false by v. Subsequently, being backed by column a, for each variable p, row p or row $\neg p$, as the case may be, eliminates row $\neg(p \wedge \neg p)$. Next, a itself is eliminated by b, removing the blocks at a on C_i for each i with $1 \leq i \leq k$. Having assumed φ to be satisfiable, for each clause C_i there still is a column λ_i^j present. Backed by this column, row C_i can now be eliminated by row (λ_i^j, i). All rows $\neg(p \wedge \neg p)$ for a variable p and C_i for $1 \leq i \leq k$ being removed, column b eliminates all remaining columns, thus reaching subgame $(\Gamma'_\varphi, X'_1, X'_2)$.

For the opposite direction assume that $(\Gamma'_\varphi, X'_1, X'_2)$ is reachable from (Γ, X_1, X_2) and let σ be the witnessing elimination sequence. Now observe that for each variable p occurring in φ row $\neg(p \wedge \neg p)$ is eliminated. Recall that this is only possible when at least one of the columns p and $\neg p$ is eliminated first and when column a is still present to back the elimination. Also, for each $1 \leq i \leq k$ row C_i is eliminated in σ. This elimination, however, is only possible by some row (λ_i^j, i) backed by column λ_i^j, and only when column a is no longer there to block it. Now define a valuation v^* such that v^* satisfies all literals λ_i^j represented by columns that are still present at the point that column a is eliminated in σ. It follows that v^* is well-defined and also satisfies φ. □

By definition, solvability is a special case of subgame reachability, which Theorem 1 shows to be intractable in constant-sum games. For single-winner games, i.e., constant-sum games consisting only of the outcomes $(0, 1)$ and $(1, 0)$, this problem is tractable [5]. Whether solvability is tractable in general constant-sum games, however, remains an open question.

4.2 Eliminability

The failure of IWD being order independent, can be rephrased as that, for elimination sequences $\sigma = (\sigma_1, \ldots, \sigma_k)$ and actions d, σ being valid in a game Γ does not generally imply that σ is still valid in $\Gamma(d)$ (or that $(\sigma_1, \ldots, \sigma_{k-1}, \sigma_{k+1}, \ldots, \sigma_n)$ is, if $d = \sigma_k$). The problem is that there may be i with $1 \leq i \leq n$ such that action d is the only action in $\Gamma(\sigma_1, \ldots, \sigma_{i-1})$ that backs the elimination of σ_i. Eliminating d too early may thus render σ_i uneliminable. For two-player constant-sum games, we find, however, that under particular circumstances and for a particular type of elimination sequence, which we will call *essential*, one can carry out the elimination of d earlier and still be able to eliminate all of the actions $\sigma_1, \ldots, \sigma_n$, provided one is prepared to postpone the elimination of some of them. This observation forms the basis of Theorem 2, proving that the eliminability problem for two-player constant-sum games can be solved efficiently.

Fix a game Γ and let $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\delta = (\delta_1, \ldots, \delta_n)$ be sequences of actions. We say that σ *is valid in* Γ *with respect to* δ if for each i with $1 \leq i \leq n$, action δ_i dominates σ_i in $\Gamma(\sigma_1, \ldots, \sigma_{i-1})$. This implies that $\delta_i \notin \{\sigma_1, \ldots, \sigma_i\}$. Given σ and δ we also define for each i with $1 \leq i \leq k$,

$$B_i(\delta, \sigma) = \{(\delta_j, \sigma_j) \colon \sigma_i \text{ blocks the elimination of } \sigma_j \text{ by } \delta_j \text{ in } \Gamma\}.$$

Observe that $j > i$ for all $(\delta_j, \sigma_j) \in B_i(\delta, \sigma)$, if σ is valid in Γ with respect to δ. Also, if $B_i(\delta, \sigma) = \emptyset$ for some i with $1 \leq i \leq n$ the elimination sequence $(\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$ is also valid in Γ. We say that σ is *essential* if

$B_i(\delta,\sigma) \neq \emptyset$ for all i with $1 \leq i < n$. An action σ_i is said to an *obstacle* in σ with respect to δ if δ_i does dominate σ_i in $\Gamma(\sigma_1,\ldots,\sigma_{i-1})$. The set of obstacles of σ with respect to δ we denote by $O(\delta,\sigma)$. Finally, for $\sigma = (\sigma_1,\ldots,\sigma_n)$ a sequence of actions and $1 \leq i < j \leq n$, we write

$$\sigma^{i\to j} = (\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_j,\sigma_i,\sigma_{j+1},\ldots,\sigma_n).$$

Thus, $\sigma^{i\to j}$ is exactly like σ with the only difference that σ_i moved to the position directly behind σ_j. We now have the following useful lemma, which specifies sufficient conditions under which the elimination of an action can be delayed without producing new obstacles apart from the action itself.

Lemma 3. *Let $\sigma = (\sigma_1,\ldots,\sigma_n)$ and $\delta = (\delta_1,\ldots,\delta_n)$ be sequences of actions of a game $\Gamma = (A_1, A_2, u)$. Fix an i with $1 \leq i \leq n$ and let m be the smallest index k with $i < k \leq n$ such that $(\delta_k,\sigma_k) \in B_i(\delta,\sigma)$. Then, for all j with $i \leq j < m$, $O(\delta^{i\to j},\sigma^{i\to j}) \subseteq O(\sigma,\delta) \cup \{\sigma_i\}$.*

Proof. Consider an arbitrary σ_k with $1 \leq k \leq n$ and $k \neq i$. Assume that σ_k is no obstacle in σ with respect to δ, i.e., σ_k be dominated by δ_k at some action x in $\Gamma(\sigma_1,\ldots,\sigma_{k-1})$. The only interesting case is $i < k \leq j$, as otherwise $\Gamma(\sigma_1,\ldots,\sigma_{k-1}) = \Gamma(\sigma_1^{i\to j},\ldots,\sigma_{k-1}^{i\to j})$. Hence, $\sigma_k = \sigma_{k-1}^{i\to j}$ and $(\delta_k,\sigma_k) \notin B_i(\delta,\sigma)$. Observe that in $\Gamma(\sigma_1^{i\to j},\ldots,\sigma_{k-1}^{i\to j})$, action x still backs the elimination of σ_k by δ_k. As $(\delta_k,\sigma_k) \notin B_i(\delta,\sigma)$, σ_i does not block this elimination, neither do any other actions in $\Gamma(\sigma_1^{i\to j},\ldots,\sigma_{k-1}^{i\to j})$. Therefore, σ_k is no obstacle in $\sigma^{i\to j}$. □

One corollary of Lemma 3 is that, under the conditions specified, $\sigma^{i\to j}$ is valid in Γ if σ is. Furthermore, if an obstacle σ_i in σ is moved to a position j where it is no longer an obstacle, and j is smaller than the smallest index with $(\delta_k,\sigma_k) \in B_i(\delta,\sigma)$ but greater than i, the number of obstacles strictly decreases, i.e., $|O(\delta^{i\to j},\sigma^{i\to j})| < |O(\delta,\sigma)|$. We now have the following lemma.

Lemma 4. *Let $\Gamma = (A_1, A_2, u)$ be a constant-sum game. Let a, b and c be distinct actions in $A_1 \cup A_2$ and σ a valid elimination sequence in Γ containing neither a, b nor c. Then, if a is eliminable by b at c in Γ, a is still eliminable by b at c in $\Gamma(\sigma)$.*

Proof. Let d be an action distinct from a, b and c that is dominated by some action x in Γ. It suffices to prove the following:

If a is eliminable by b at c in Γ, then a is eliminable by b at c in $\Gamma(d)$.

The lemma then follows by a straightforward induction.

Assume a is eliminable by b at c in Γ. Accordingly, there are sequences $\sigma = (\sigma_1,\ldots,\sigma_n)$ and $\delta = (\delta_1,\ldots,\delta_n)$ in Γ such that σ is valid with respect to δ, $c \notin \{\sigma_1,\ldots,\sigma_n\}$, $\sigma_n = a$, $\delta_n = b$ and a being dominated by b at c in $\Gamma(\sigma_1,\ldots,\sigma_{n-1})$. Without loss of generality we may assume that σ is essential. We also make the following observations. First, we may assume without loss of generality that $d \neq \delta_i$ for all $1 \leq i \leq n$. Let d is dominated by action x in Γ and set for each i with $1 \leq i \leq n$

$$x_i = \begin{cases} x & \text{if } i = 1, \\ \delta_{i-1} & \text{if } x_{i-1} = \sigma_{i-1}, \\ x_{i-1} & \text{otherwise.} \end{cases}$$

Then, by transitivity of dominance, for each i with $1 \leq i \leq n$, if d dominates σ_i so does x_i and thus can go proxy for d. Second, σ_n is no obstacle in $(d, \sigma_1, \dots, \sigma_n)$. To appreciate this, observe that in $\Gamma(\sigma_1, \dots, \sigma_{n-1})$ there are no actions blocking the elimination of $\sigma_n = a$ by $\delta_n = b$, so neither are there any such actions in $\Gamma(d, \sigma_1, \dots, \sigma_{n-1})$. Moreover, c still backs the elimination of $\sigma_n = a$ by $\delta_n = b$. Hence, $\sigma_n = a$ is dominated by $\delta_n = b$ at c in $\Gamma(d, \sigma_1, \dots, \sigma_{n-1})$.

We consider the case in which $d \notin \{\sigma_1, \dots, \sigma_n\}$; apart from some tedious details, the case in which $d \in \{\sigma_1, \dots, \sigma_n\}$ runs along analogous lines. We show by induction on the number of obstacles in $(d, \sigma_1, \dots, \sigma_n)$ with respect to $(x, \delta_1, \dots, \delta_n)$ that there is a sequence $\tau = (\tau_1, \dots, \tau_n)$ that is valid in $\Gamma(d)$ and, moreover, is such that b dominates a at c in $\Gamma(d, \tau_1, \dots, \tau_{n-1})$.

If $(d, \sigma_1, \dots, \sigma_n)$ contains no obstacles, σ is obviously valid in $\Gamma(d)$ and can be taken as a witness for τ. So assume $(d, \sigma_1, \dots, \sigma_n)$ contains more than one obstacle and consider the obstacle σ_i with the smallest index i. Because σ_n is no obstacle, $\sigma_i \neq \sigma_n$. Consequently, σ being both valid and essential, there is some smallest index $j > i$ such that $(\delta_j, \sigma_j) \in B_i(\delta, \sigma)$. Without loss of generality assume that $\sigma_i \in A_1$ and $\delta_j, \sigma_j \in A_2$. Then, $u_2(\sigma_i, \delta_j) < u_2(\sigma_i, \sigma_j)$ and, by Γ being constant-sum, $u_1(\sigma_i, \delta_j) > u_1(\sigma_i, \sigma_j)$. Because δ_i dominates σ_i in $\Gamma(\sigma_1, \dots, \sigma_{i-1})$ but not in $\Gamma(d, \sigma_1, \dots, \sigma_{i-1})$ and $\delta_i \neq d$, it follows that the only dominance of δ_i over σ_i in $\Gamma(\sigma_1, \dots, \sigma_{i-1})$ is at d. Consequently, $u_1(\sigma_i, x) = u_1(\delta_i, x)$ for all $x \in A_2 \setminus \{d, \sigma_1, \dots, \sigma_{n-1}\}$. In particular, $u_1(\sigma_i, \delta_j) = u_1(\delta_i, \delta_j)$ and $u_1(\sigma_i, \sigma_j) = u_1(\delta_i, \sigma_j)$. With Γ being constant-sum, it follows that $u_2(\delta_i, \delta_j) < u_2(\delta_i, \sigma_j)$, i.e., δ_i blocks the elimination of σ_j by δ_j in Γ. Accordingly, δ_i is eliminated before σ_j in σ, i.e., $\delta_i = \sigma_{i_1}$ for some $i < i_1 < j$ and $(\delta_j, \sigma_j) \in B_{i_1}(\delta, \sigma)$. Repeating this argument, there are $i = i_0 < i_1 < \cdots < i_m < j$ such that $\sigma_{i_k} = \delta_{i_{k-1}}$, $(\delta_j, \sigma_j) \in B_{i_k}(\delta, \sigma)$ for $1 \leq k \leq m$, and $u_2(\delta_{i_m}, \delta_j) \geq u_2(\delta_{i_m}, \sigma_j)$. The latter because otherwise δ_j would not dominate σ_j in $\Gamma(\sigma_1, \dots, \sigma_{j-1})$. As Γ is constant-sum, moreover, $u_1(\delta_{i_m}, \sigma_j) \geq u_1(\delta_{i_m}, \delta_j)$. By transitivity of dominance, then, $u_1(\delta_{i_m}, \delta_j) \geq u_1(\sigma_i, \delta_j)$. The situation is depicted in Figure 2. It follows that $u_1(\delta_{i_m}, \sigma_j) > u(\sigma_i, \sigma_j)$. As a consequence, δ_{i_m} dominates σ_i at σ_j in $\Gamma(d, \sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_{i_m})$.

Now consider the elimination sequence $\sigma^{i \to i_m} = (\sigma_1^{i \to i_m}, \dots, \sigma_n^{i \to i_m})$. Recall that $i_m < j$. By choice of j and Lemma 3, the sequence $\sigma^{i \to i_m}$ is valid in Γ and $(d, \sigma_1^{i \to i_m}, \dots, \sigma_n^{i \to i_m})$ contains fewer obstacles with respect to $(x, \delta_1^{i \to i_m}, \dots, \delta_n^{i \to i_m})$ than $(d, \sigma_1, \dots, \sigma_n)$ does with respect to $(x, \delta_1, \dots, \delta_n)$. By virtue of the induction hypothesis, we may conclude that there is an elimination sequence $\tau = (\tau_1, \dots, \tau_n)$ that is valid in $\Gamma(d)$ and is such that b dominates a at c in $\Gamma(d, \tau_1, \dots, \tau_{n-1})$ □

	d	δ_j		σ_j
δ_{i_m}		$u_1(\delta_{i_m}, \delta_j)$	$\leq$	$u_1(\delta_{i_m}, \sigma_j)$
		$\rotatebox{90}{$\leq$}$		$\rotatebox{90}{$\leq$}$
$\delta_{i_{m-1}} = \sigma_{i_m}$		$u_1(\delta_{i_{m-1}}, \delta_j)$	$>$	$u_1(\delta_{i_{m-1}}, \delta_j)$
		$\rotatebox{90}{$\leq$}$		$\rotatebox{90}{$\leq$}$
$\vdots$		$\vdots$	$\vdots$	$\vdots$
		$\rotatebox{90}{$\leq$}$		$\rotatebox{90}{$\leq$}$
$\delta_i = \sigma_{i_1}$	$u_1(\delta_i, d)$	$u_1(\delta_i, \delta_j)$	$>$	$u_1(\delta_i, \sigma_j)$
	$\vee$	$\parallel$		$\parallel$
σ_i	$u_1(\sigma_i, d)$	$u_1(\sigma_i, \delta_j)$	$>$	$u_1(\sigma_i, \sigma_j)$

Fig. 2. Diagram illustrating the proof of Lemma 4

Intuitively, Lemma 4 says that, if one wishes to eliminate a particular action a by b backed by c, one can proceed greedily and eliminate any action whenever possible. One just has to be careful not to eliminate the actions b and c before a is eliminated. On the basis of this observation, we obtain the following result.

Theorem 2. *Given a two-player constant-sum game Γ, deciding whether a particular action a is eliminable can be decided in polynomial time.*

Proof. Without loss of generality we may assume that $a \in A_1$. Now consider the algorithm that performs the following steps:

1. Compose a list $(b_1, c_1), \ldots, (b_k, c_k)$ of all actions $b_i \in A_1$ and $c_i \in A_2$ such that c_i backs the elimination of a by b_i.
2. For each i with $1 \leq i \leq k$ arbitrarily eliminate any actions *distinct from b_i and c_i* until no more eliminations are possible. Let $\sigma^i = (\sigma^i_1, \ldots, \sigma^i_{m_i})$ denote the resulting valid elimination sequence.
3. If for some i with $1 \leq i \leq k$, action a is eliminated in σ_i, i.e., $a \in \{\sigma^i_1, \ldots, \sigma^i_{m_i}\}$, output "yes", otherwise "no".

Obviously, this algorithm runs in polynomial time. Moreover, soundness follows from Lemma 4. □

5 Win-Lose Games

Conitzer and Sandholm [6] show that subgame reachability and eliminability are NP-complete in *win-lose* games, i.e., games which only allow the outcomes $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. As both win-lose and constant-sum games generalize single-winner games, it is interesting to compare their results with the ones for constant-sum games in the previous section. We show that Conitzer and Sandholm's results even hold for win-lose games with at most one winner, i.e., for games with $(0,0)$, $(0,1)$, $(1,0)$ as only outcomes.

Theorem 1 established that subgame reachability is NP-complete for games with outcomes in $\{(0,1),(1,0)\}$. This obviously implies that this problem is also NP-complete when additionally allowing outcome $(0,0)$.

It turns out that eliminability is also NP-complete for win-lose games with at most one winner. The proof is a reduction from *3SAT* and involves a modification of the construction used in the proof of Theorem 1.

Theorem 3. *Given a game Γ with outcomes in $\{(0,0),(0,1),(1,0)\}$, deciding whether a particular action is eliminable is NP-complete.*

Conitzer and Sandholm [6] reduced eliminability to solvability in win-lose games to establish the computational intractability of the latter problem. Their construction hinges on the presence of the outcome $(1,1)$. Our reduction for the more restricted class of games without $(1,1)$ as an outcome is directly from *3SAT* and exploits the internal structure of the construction used in the proof of Theorem 3.

Theorem 4. *Deciding whether a game Γ with outcomes in $\{(0,0),(0,1),(1,0)\}$ is solvable is NP-complete.*

6 Conclusion

We investigated the computational complexity of iterated weak dominance in two-player constant-sum games. In particular, we showed that deciding whether an action is eliminable is feasible in polynomial time whereas deciding whether a given subgame is reachable is NP-complete. Furthermore, we proved that typical problems associated with iterated dominance in win-lose games *with at most one winner* are NP-complete.

Conitzer and Sandholm [6] have shown that in win-lose games an action is dominated by a mixed strategy if and only if it is dominated by a pure strategy. Thus, although our analysis has been restricted to dominance by pure strategies, it is readily appreciated that all of our results, apart from Theorem 2, immediately extend to dominance by mixed strategies.

A solution concept related to weak dominance is *very weak* dominance, which also allows a player to eliminate one of two actions between which he is completely indifferent. Knuth et al. [9] have shown that iterated very weak dominance in constant-sum games always results in isomorphic subgames, and that deciding whether an action can be eliminated in this context is P-complete. It is not very hard to see that all problems considered in this paper are tractable when replacing weak dominance with very weak dominance. In a similar spirit, Marx and Swinkels [12] have shown that, in constant-sum games, all subgames that are reachable via iterated *weak* dominance (subject to no more eliminations being possible) are payoff-equivalent in the sense that they are the same up to the addition or removal of identical actions. However, this property does not imply any of our results because it does not discriminate between actions that yield identical payoffs for *some* reachable subgame. The conceptual difference between this paper and the above mentioned work is linked to the question whether one is interested in *action profiles* or *payoff profiles* as "solutions" of a game, or, more generally, whether one champions a prescriptive or a descriptive interpretation of game theory. It may be argued that the computational gap between both concepts is of particular interest in this context.

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A Proofs

Lemma 1. *For each regionalized game (Γ, X_1, X_2) with $\Gamma = (A_1, A_2, u)$, there is a game $\Gamma' = (A'_1, A'_2, u')$ computable in polynomial time such that the valid elimination sequences of Γ' and (Γ, X_1, X_2) coincide:*

$$\{\Sigma \colon \Sigma \text{ a valid sequence in } \Gamma'\} = \{\Sigma \colon \Sigma \text{ a valid sequence in } (\Gamma, X_1, X_2)\}.$$

Moreover, $u'(a, b) \in \{(0,1), (1,0)\}$ for all $a \in A'_1 \setminus A_1$ and $b \in A'_2 \setminus A_2$.

					X_1						
		a_2^1	$\cdots$	a_2^m	x_2^1	$\cdots$	x_2^l	y_2^1	y_2^2	y_2^3	y_2^4
	a_1^1	$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$(0,1)$	$(1,0)$	$(0,1)$	$(1,0)$
	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	a_1^n	$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$(0,1)$	$(1,0)$	$(0,1)$	$(1,0)$
X_2	x_1^1	$\cdot$	$\cdots$	$\cdot$	$(1,0)$	$\cdots$	$(0,1)$	$(1,0)$	$(0,1)$	$(1,0)$	$(0,1)$
X_2	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X_2	x_1^k	$\cdot$	$\cdots$	$\cdot$	$(0,1)$	$\cdots$	$(1,0)$	$(1,0)$	$(0,1)$	$(1,0)$	$(0,1)$
	y_1^1	$(1,0)$	$\cdots$	$(1,0)$	$(0,1)$	$\cdots$	$(0,1)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$
	y_1^2	$(0,1)$	$\cdots$	$(0,1)$	$(1,0)$	$\cdots$	$(1,0)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$
	y_1^3	$(1,0)$	$\cdots$	$(1,0)$	$(0,1)$	$\cdots$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$
	y_1^4	$(0,1)$	$\cdots$	$(0,1)$	$(1,0)$	$\cdots$	$(1,0)$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$

Fig. 3. Construction used in the proof of Lemma 1

Proof. The game Γ' is constructed from Γ by adding actions that impose the same restrictions on the elimination of actions as the regions did in (Γ, X_1, X_2). More ancillary actions are added, ensuring that all elimination sequences that are valid in (Γ, X_1, X_2) are still valid in Γ' while no new valid elimination sequences are created.

Formally, define $\Gamma' = (A'_1, A'_2, u')$ as the game with $A'_1 = A_1 \cup X_2 \cup \{y_1^1, y_1^2, y_1^3, y_1^4\}$ and, analogously, $A'_2 = A_2 \cup X_1 \cup \{y_2^1, y_2^2, y_2^3, y_2^4\}$. Furthermore, define the utility function u' such that $u'(a, b) = u(a, b)$ for all $(a, b) \in A_1 \times A_2$. For every $(a, x) \in A_1 \times X_1$ and $(x, b) \in X_2 \times A_2$, let

$$u'(a, x) = \begin{cases} (1,0) & \text{if } a \in x, \\ (0,1) & \text{otherwise;} \end{cases} \quad u'(x, b) = \begin{cases} (0,1) & \text{if } b \in x, \\ (1,0) & \text{otherwise.} \end{cases}$$

Without loss of generality we may assume that $|X_1| = |X_2|$, as we can always introduce copies of actions to the game and none of the actions in $X_1 \cup X_2$ will be eliminated at any valid elimination sequence of Γ'. For all $(x_1^i, x_2^j) \in X_1 \times X_2$, let

$$u'(x_1^i, x_2^j) = \begin{cases} (1,0) & \text{if } i = j, \\ (0,1) & \text{otherwise.} \end{cases}$$

The payoffs for the remaining action profiles are depicted in Figure 3. Obviously, Γ' can be obtained from (Γ, X_1, X_2) in polynomial time.

The utility function u' is chosen so as to assure that in Γ' none of the additional actions in $X_2 \cup \{y_1^1, y_1^2, y_1^3, y_1^4\}$ dominate or are dominated by any other action in A_1' and, similarly, that none of the additional actions in $X_1 \cup \{y_2^1, y_2^2, y_2^3, y_2^4\}$ dominate or are dominated by some other action in A_2'. For example, rows y_1^1 and y_1^2 do not dominate each other because of the blocks at columns a_2^1 and x_2^1. Moreover, for each $a' \in A_1$ and each $z \in X_2 \cup \{y_1^1, y_1^2, y_1^3, y_1^4\}$ we have $u(a, z) = u(b, z)$. Hence, the presence of the actions in $X_2 \cup \{y_1^1, y_1^2, y_1^3, y_1^4\}$ do not hamper eliminations possible in Γ, neither do they enable such eliminations that were not possible before. An analogous statement holds for columns in A_2. Now, consider actions $a, b \in A_1 \cup A_2$ such that a dominates b in Γ but not in (Γ, X_1, X_2), because a and b are in different regions x_i and x_j, respectively. Then, this elimination is blocked by action x_j in Γ'. Finally, observe that the elimination of actions in $A_1 \cup A_2$ does not affect the overall structure of the game and the argument can be repeated. □

Lemma 2. *Let Γ be a two-player constant-sum game and $\Sigma = (\Sigma_1, \ldots, \Sigma_m)$ a valid elimination sequence. Then, there is a simple elimination sequence $\sigma = (\sigma_1, \ldots, \sigma_k)$ with $\{\sigma_1, \ldots, \sigma_m\} = \Sigma_1 \cup \cdots \cup \Sigma_m$ that is also valid in Γ.*

Proof. Let X be a non-empty subset of A. It suffices to show that validity of X as an elimination sequence for Γ implies the existence of some $x \in X$, such that the sequence $(X \setminus \{x\}, \{x\})$ is valid in Γ as well.

Assume for contradiction that this was not the case. Consider an arbitrary $x \in X$ and assume without loss of generality that $x \in A_1$. Then, x is dominated by some $x' \in A_1$ at some $y \in A_2$ in Γ, i.e., $u_1(x', y) > u_1(x, y)$. By transitivity of dominance, we may further assume without loss of generality that $x' \notin X$. If $y \notin X$, then $(X \setminus \{x\}, \{x\})$ is valid in Γ, a contradiction. To see this, observe that x' dominates x in Γ, and as such no actions block this elimination in Γ or in $\Gamma(X \setminus \{x\})$. Moreover, y still backs the elimination of x by x' in $\Gamma(X \setminus \{x\})$.

If otherwise $y \in X$, there must be some $y' \in A_2$ dominating y. By transitivity of dominance we may assume that $y' \notin X$. Having assumed that $(X \setminus \{y\})$ is not valid in Γ, we have that $u_2(x', y') \leq u_2(x', y)$. Thus, since Γ is a constant-sum game, $u_1(x', y') \geq u_1(x', y)$. As x' dominates x, we also have $u_1(x', y') \geq u_1(x, y')$. Moreover, since $(X \setminus \{x\}, \{x\})$ is not valid in Γ, $u_1(x', y') \leq u_1(x, y')$ and thus $u_1(x', y') = u_1(x, y')$. This situation is illustrated in Figure 4.

It now follows that $u_1(x, y') > u_1(x, y)$ and thus $u_2(x, y') < u_2(x, y)$, contradicting the assumption that y' dominates y in Γ. □

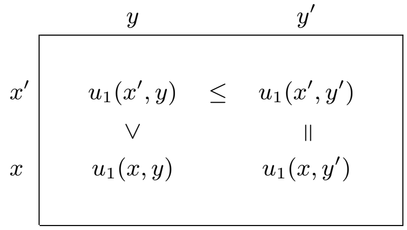

Fig. 4. Diagram illustrating the proof of Lemma 2

Theorem 3. *Given a game Γ with outcomes in $\{(0,0),(0,1),(1,0)\}$, deciding whether a particular action is eliminable is NP-complete.*

Proof. Membership in NP is obvious. The proof of hardness proceeds by a reduction from *3SAT*. By virtue of Lemma 1 it suffices to prove this for regionalized games. For each *3CNF* φ we modify the construction $(\Gamma_\varphi, X_1, X_2)$ with $\Gamma_\varphi = (A_1, A_2, u)$ as in the proof of Theorem 1 for constant-sum games. Observe that the latter only involved the outcomes $(0,1)$ and $(1,0)$. Define the regionalized game $(\Gamma'_\varphi, X'_1, X'_2)$ such that $A'_1 = A_1$, $A'_2 = A_2 \cup \{c, d^*\}$, $X'_1 = X_1$ and $X'_2 = X_2 \cup \{\{c, d^*\}\}$. The utility function u' extends u, i.e., $u'(a,b) = u(a,b)$ for all $a \in A_1$ and $b \in A_2$. The payoffs assigned by u' in columns c and d^* are summarized in the table below. Also see Figure 5 for an example.[1]

	c	d^*
p	$(0,0)$	$(0,0)$
$\neg p$	$(0,0)$	$(0,0)$
$\neg(p \wedge \neg p)$	$(0,0)$	$(0,1)$

	c	d^*
(λ_i^1, i)	$(0,0)$	$(0,0)$
(λ_i^2, i)	$(0,0)$	$(0,0)$
(λ_i^3, i)	$(0,0)$	$(0,0)$
C_i	$(0,0)$	$(0,1)$

	c	d^*
e	$(0,1)$	$(0,0)$

Observe that for all rows $x, x' \in A'_1$, we have $u_1(x,c) = u_1(x',c)$ and $u_1(x,d^*) = u_1(x',d^*)$. Accordingly, the additional columns c and d^* do not back or block any eliminations of rows that are possible in $(\Gamma_\varphi, X_1, X_2)$. Because c and d, moreover, constitute a separate region, they are not eliminable by any columns in A_2, neither are the latter eliminable by the former. Finally, observe that column d^* is dominated by c at e if and only for each propositional variable p and each clause C_i the rows $\neg(p \wedge \neg p)$ and C_i are eliminated. Hence, by virtue of an argument analogous to the one that proved Theorem 1, we find that column d^* is eliminable if and only if φ is satisfiable, giving us the result. □

[1] By setting $u'_1(x,c) = u'_1(x,d^*) = 1$ instead, one obtains a construction proving the intractability of the eliminability problem for games with outcomes in $(0,1)$, $(1,0)$ and $(1,1)$.

	p	$\neg p$	q	$\neg q$	r	$\neg r$	a	b	c	d^*
p	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg p$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg(p \wedge \neg p)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,1)$
q	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg q$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg(q \wedge \neg q)$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,1)$
r	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg r$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg(r \wedge \neg r)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,1)$
p	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
q	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg r$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
$p \vee q \vee \neg r$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,1)$
$\neg p$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
q	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
r	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg p \vee q \vee r$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,1)$
$\neg p$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg q$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg r$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$
$\neg p \vee \neg q \vee \neg r$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,1)$
e	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,0)$

Fig. 5. Example of the construction used in the proof of Theorem 3. This is the case for the formula $(p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$.

	a_2^1	$\cdots$	a_2^m	c	d^*	f	g^*	x_2^1	$\cdots$	y_2^4	z_2^1	z_2^2	z_2^3
a_1^1	$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$(0,1)$	$(0,1)$	$(0,1)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_1^n	$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$(0,1)$	$(0,1)$	$(0,1)$
e	$\cdot$	$\cdots$	$\cdot$	$(0,1)$	$(0,0)$	$(0,1)$	$(0,0)$	$\cdot$	$\cdots$	$\cdot$	$(0,1)$	$(0,1)$	$(0,1)$
x_1^1	$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$(0,1)$	$(0,1)$	$(0,1)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_1^4	$\cdot$	$\cdots$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$(0,1)$	$(0,1)$	$(0,1)$
z_1^1	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,0)$
z_1^2	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$
z_1^3	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(1,0)$	$(0,1)$
z_1^4	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,1)$	$(1,0)$

Fig. 6. Construction used in the proof of Theorem 4

Theorem 4. *Deciding whether a game Γ with outcomes in $\{(0,0),(0,1),(1,0)\}$ is solvable is NP-complete.*

Proof. Membership in NP is straightforward. Hardness is proved by a reduction from *3SAT*. Thus, let φ be a *3CNF* formula and $(\Gamma'_\varphi, X'_1, X'_2)$ with $\Gamma'_\varphi = (A'_1, A'_2, u')$ the regionalized game with outcomes in $\{(0,0),(0,1),(1,0)\}$ for φ as in Theorem 3, be it that we have additional copies f and g^* of the columns c and d^*, respectively. Let $\{f, g^*\}$, moreover, constitute a separate region. Thus, A'_1 and X'_1 are as before, however, for the column player we have

$$A'_2 = \{p, \neg p \colon p \text{ a variable in } \varphi\} \cup \{a, b, c, d^*, f, g^*\},$$
$$X'_2 = \{\{p, \neg p \colon p\text{a variable in } \varphi\} \cup \{a, b\}, \{c, d^*\}, \{f, g^*\}\},$$

and for each row $x \in A'_1$, $u'(x, f) = u'(x, c)$ and $u'(x, g^*) = u'(x, d^*)$. Clearly, φ is satisfiable if and only if both d^* and g^* are eliminable (cf. Theorem 3). Now let $\Gamma''_\varphi = (A''_1, A''_2, u'')$ be defined as the non-regionalized pendant of Γ'_φ as defined in the proof of Lemma 1. We now extend Γ''_φ to a game $\Gamma'''_\varphi = (A'''_1, A'''_2, u''')$ with $A'''_1 = A''_1 \cup \{z_1^1, z_1^2, z_1^3\}$ and $A'''_2 = A''_2 \cup \{z_2^1, z_2^2, z_2^3, z_2^4\}$, assuming disjointness throughout, and $u'''(x, y) = u''(x, y)$ for all $(x, y) \in A''_1 \times A''_2$. For the remaining action profiles in $A'''_1 \times A'''_2$ the payoffs are depicted in Figure 6.

We make the following observations about the game Γ'''_φ:

(*i*) As long as the columns d^* and g^* are not eliminated, the rows and columns in $\{z_1^1, z_1^2, z_1^3\} \cup \{z_2^1, z_2^2, z_2^3, z_2^4\}$ do not dominate and are not dominated by any action in the game.

(*ii*) The rows z_1^2 and z_1^3 also back the elimination of d^* by c and z_1^4 the elimination of g^* by f. However, row e also backs the eliminations of d^* and c and the one of g^* by f. Because row e cannot is not eliminable in $\mathit{\Gamma}''_\varphi$, however, rows z_1^2, z_1^3 and z_1^4 do not enable or disable any eliminations that are not possible in $\mathit{\Gamma}''_\varphi$, as long as d^* and g^* have not been eliminated.

On the basis of observations (*i*) and (*ii*) and Theorem 3 it can now be shown that $\mathit{\Gamma}''_\varphi$ is solvable if and only if φ is satisfiable. If φ is solvable, $\mathit{\Gamma}'''_\varphi$ can be reduced to the game in which (z_1^1, z_2^1) is the only remaining action profile. □