Categories with Binding Structure

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Name Binding

- Many models of some form of bindable names
  - Presheaf models
  - Nominal Sets, domains, ...
  - Proof theory of ∇
- Similar structure for names and name-binding

Characterise the notion of bindable names.

Context

- Axioms for names and binding [Menni]
- Frameworks for substitution in the presence of variable binding [Fiore, Plotkin & Turi], [Power & Tanaka], [Gabbay & Mathijssen]
- Theory of Contexts [Honsell, Miculan & Scagnetto]
- ...

Binding Structure

Working with α-equivalence classes is the same as working with freshly named instances.

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A *category with binding structure* is a triple $(\mathbb{B}, \otimes, V)$ …

Category with finite limits

Monoidal Structure (freshness)

Object (variables)

$$\mathbb{B} = \text{Nominal Sets}$$

$$X \otimes Y = \{\langle x, y\rangle \in X \times Y \mid x\#y\}$$

$$V = \mathbb{A}$$

Binding Structure

Working with α-equivalence classes is the same as working with freshly named instances.

A *category with binding structure* is a triple $(\mathbb{B}, \otimes, V)$ such that the functor $\mathcal{M}_V^{\otimes} \colon cod \to (- \otimes V)^* cod$ with $\mathcal{M}_V^{\otimes} f = f \otimes V$ is an equivalence.

Codomain Fibration

Working with α-equivalence classes is the same as working with freshly named instances.

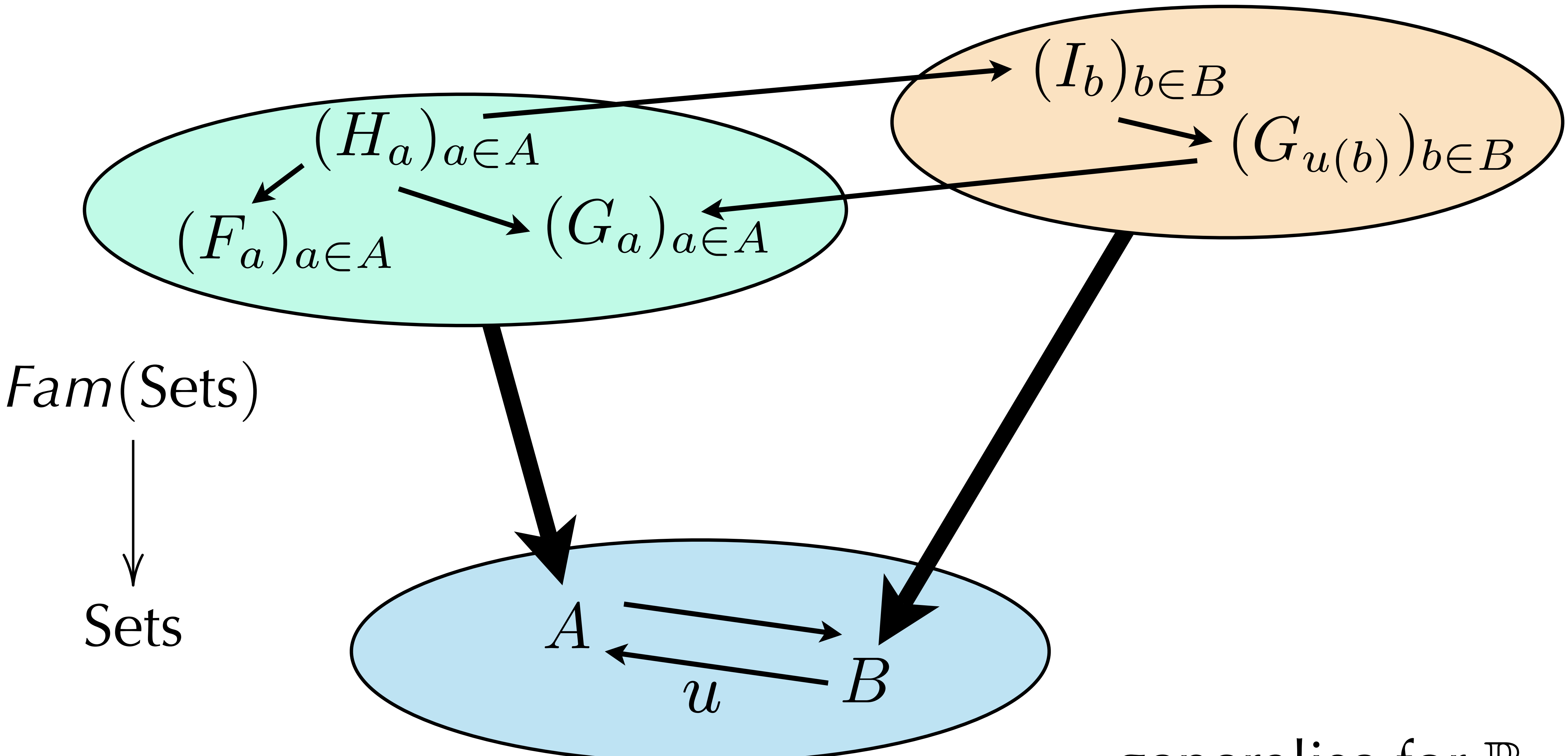
$$\begin{array}{c} \mathbb{B}^{\rightarrow} \\ \downarrow{\scriptstyle cod} \\ \mathbb{B} \end{array}$$

- Category $\mathbb{B}$ with finite limits
- Families of $\mathbb{B}$-objects indexed by $\mathbb{B}$-objects
- Dependent type theory $\Gamma \vdash A$

Other choices possible, e.g. subobject-logic

Codomain Fibration

Families of sets



...generalise for $\mathbb{B}$

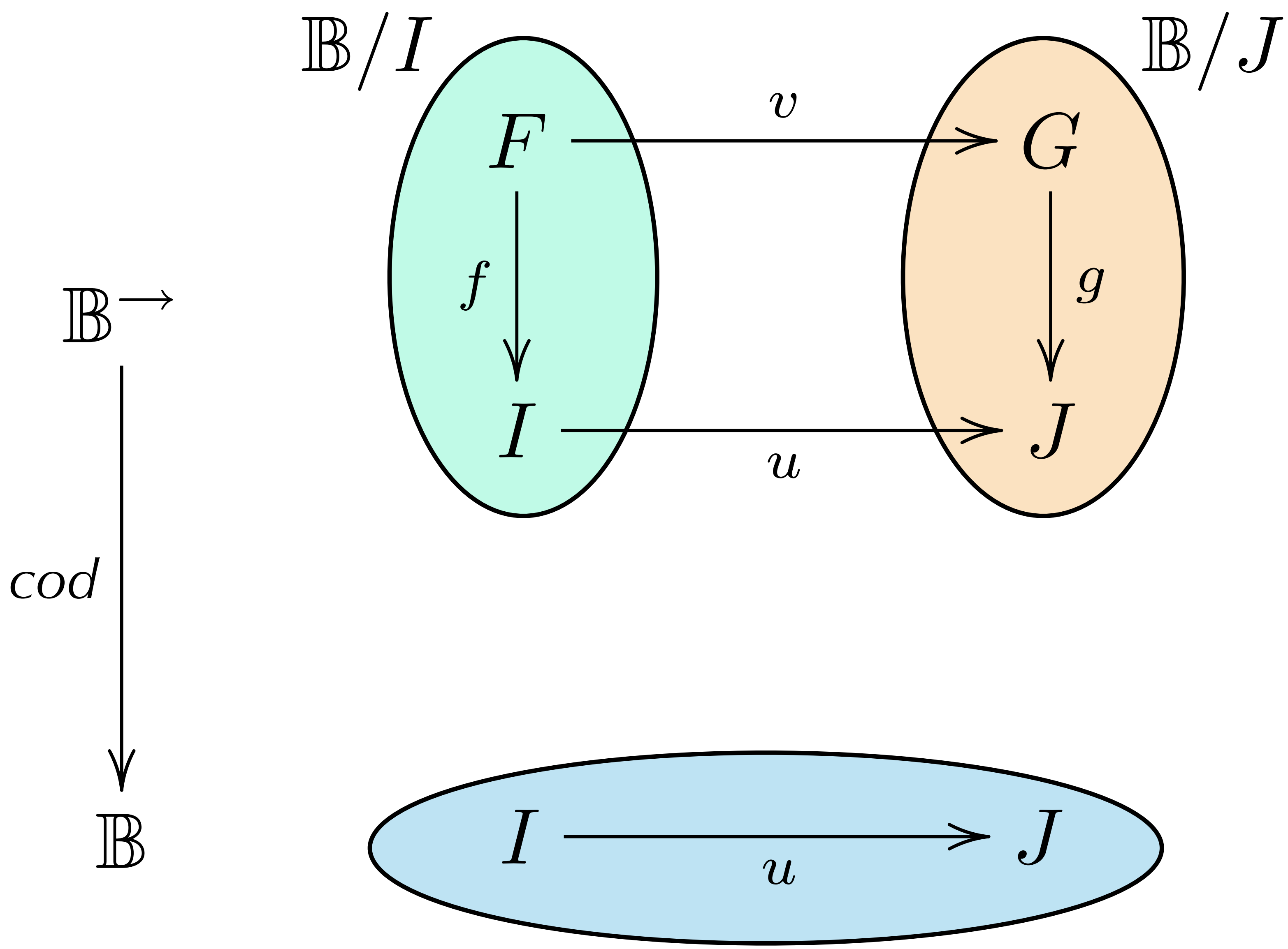
Codomain Fibration

Represent families as functions:

$$(B_a)_{a \in A} \longmapsto \left(\sum_{a \in A} B_a \xrightarrow{\pi_1} A \right)$$

$$\left(B \xrightarrow{f} A \right) \longmapsto (f^{-1}(a))_{a \in A}$$

Codomain Fibration



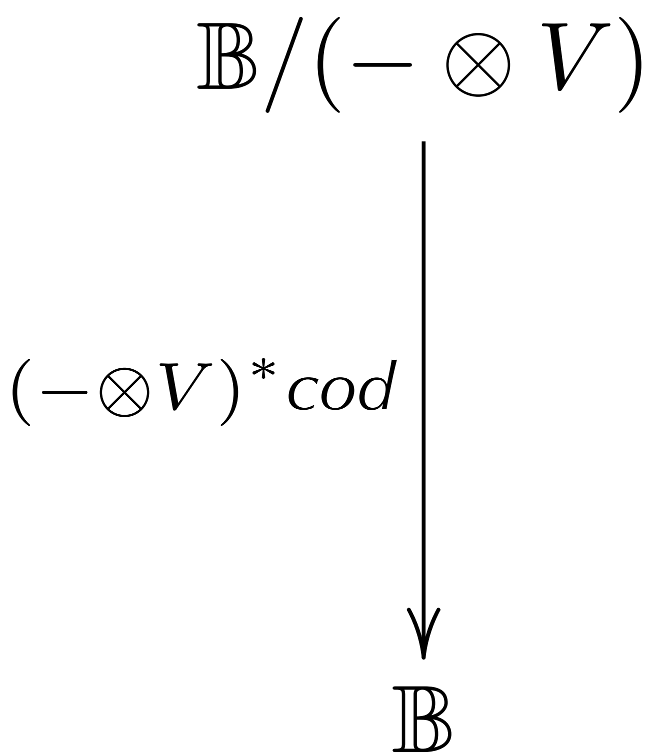
Glueing a Fresh Name

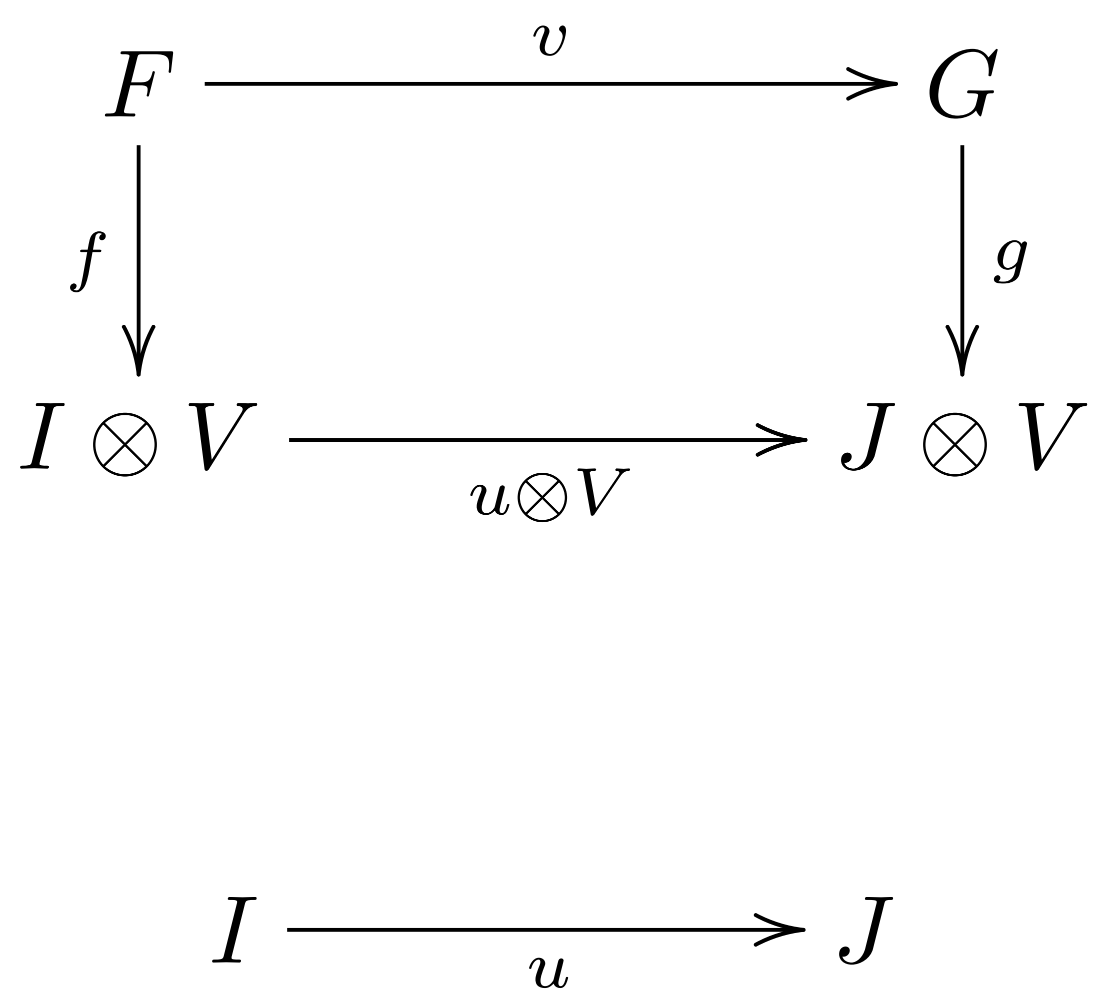
Working with α-equivalence classes is the same as working with freshly named instances.

$$\begin{array}{c} \mathbb{B}/(- \otimes V) \\ \Big\downarrow {\scriptstyle (-\otimes V)^* cod} \\ \mathbb{B} \end{array}$$

- Families of $\mathbb{B}$-objects indexed by $\mathbb{B}$-objects of the form $A \otimes V$
- Dependent type theory with judgements $\Gamma \# v{:}V \vdash A$

Glueing a Fresh Name





The Functor $\mathcal{M}_V^{\otimes}$

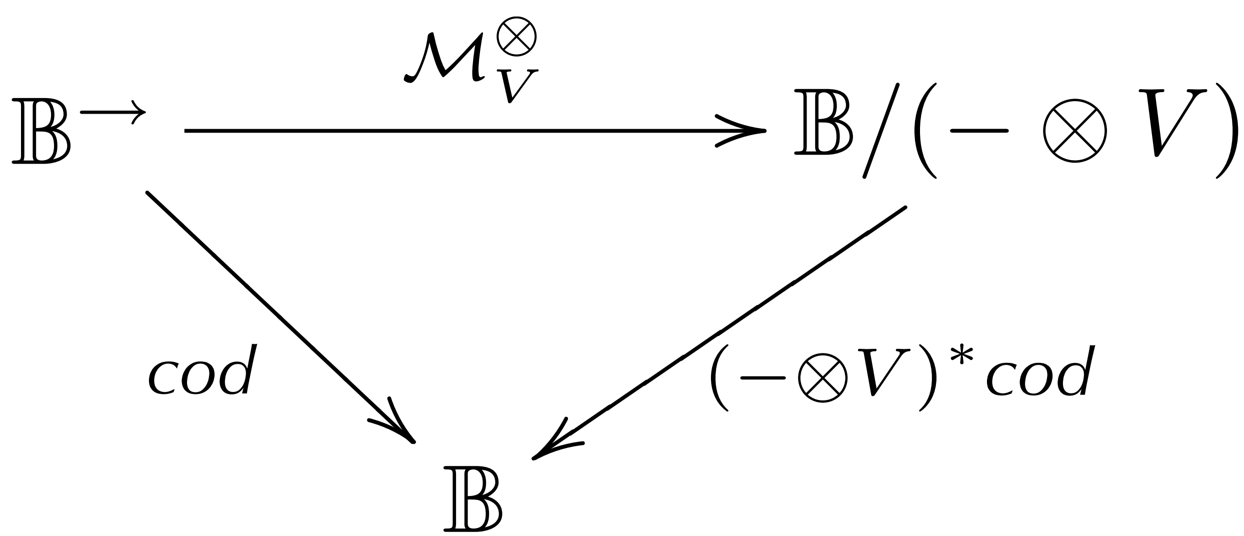
$$
\begin{array}{ccc}
X & & X \otimes V \\
{\scriptstyle f}\downarrow & \longmapsto & {\scriptstyle f \otimes V}\downarrow \\
I & & I \otimes V
\end{array}
$$

$$
\begin{array}{ccc}
\mathbb{B}^{\to} & \xrightarrow{\mathcal{M}_V^{\otimes}} & \mathbb{B}/(- \otimes V) \\
{\scriptstyle cod}\searrow & & \swarrow{\scriptstyle (-\otimes V)^* cod} \\
& \mathbb{B} &
\end{array}
$$

Binding Structure

Working with α-equivalence classes is the same as working with freshly named instances.

A *category with binding structure* is a triple $(\mathbb{B}, \otimes, V)$ such that the functor $\mathcal{M}_V^{\otimes} \colon cod \to (- \otimes V)^* cod$ with $\mathcal{M}_V^{\otimes} f = f \otimes V$ is an equivalence of fibrations.



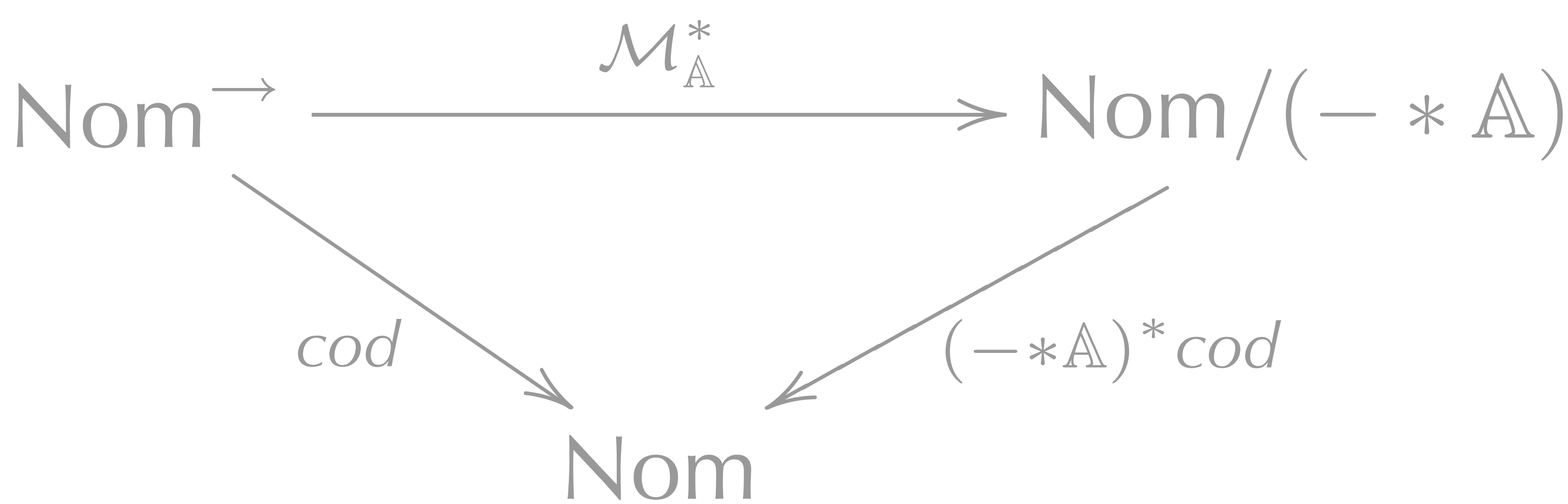
Nominal Sets

$(\mathsf{Nom}, *, \mathbb{A})$ is a category with binding structure.

$$X * Y = \{\langle x, y\rangle \in X \times Y \mid x\#y\}$$

Functor $\mathcal{M}^*_{\mathbb{A}}$:

$$(X(i))_{i\in I} \longmapsto (X(i) * \{a\})_{\langle i,a\rangle\in I*\mathbb{A}}$$



The Functor H

The following are equivalent:

1. $\mathcal{M}_V^{\otimes}$ is an equivalence of fibrations
2. There are fibred adjunctions

$$\mathrm{H} \dashv \mathcal{M}_V^{\otimes} \dashv \mathrm{H}$$

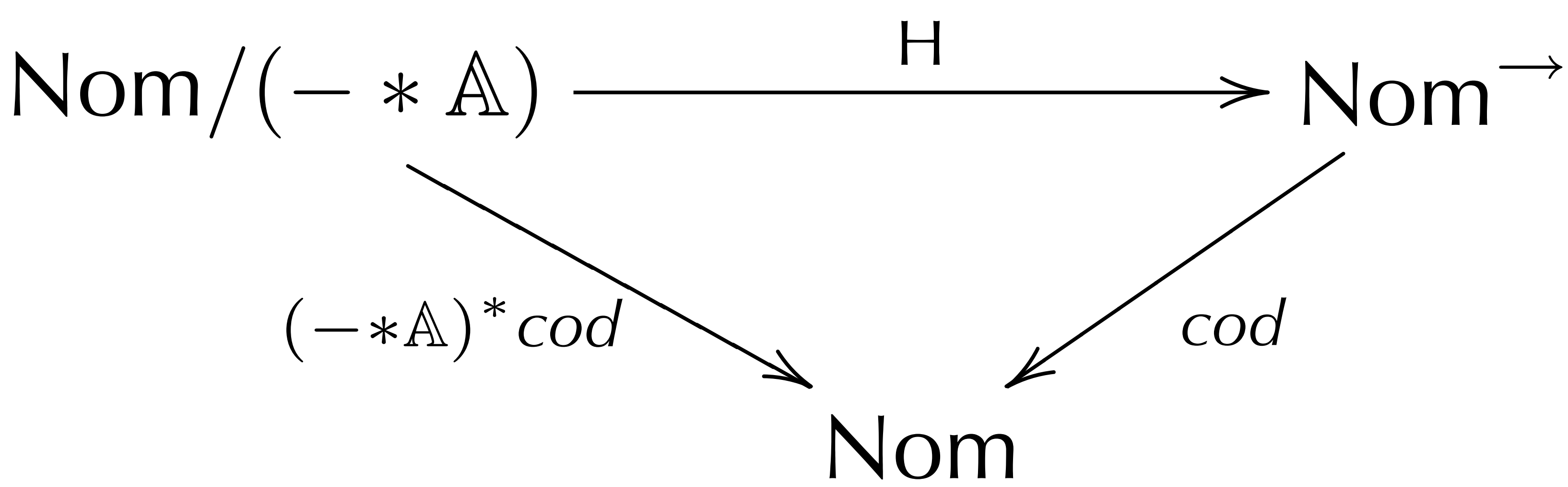
whose (co)units are related by

$$\eta^{-1} = \varepsilon' \qquad\qquad \varepsilon^{-1} = \eta'$$

H for Nominal Sets

Functor H:

$$(X(i,a))_{\langle i,a\rangle \in I * \mathbb{A}} \longmapsto (\mathsf{H}a.\, X(i,a))_{i \in I}$$



$$\mathsf{H}a.\, X(i,a) \quad = \quad \{a.x \mid a \in \mathbb{A} \wedge a\#i \wedge x \in X(i,a)\}$$

H for Nominal Sets

- H is a dependent version of Gabbay & Pitts' abstraction set
- Abstraction set
    - Nice properties, e.g.

$$[\mathbb{A}](X \Rightarrow Y) \;\cong\; ([\mathbb{A}]X) \Rightarrow ([\mathbb{A}]Y)$$

    - Operations: concretion, binding, pattern matching

How much of this structure is available in any category with binding structure?

Abstraction Type

In any category with binding structure $(\mathbb{B}, \otimes, V)$, the functor $- \otimes V$ has a right adjoint $V \multimap -$.

If $\mathbb{B}$ is cartesian closed, then $V \multimap -$ has a further right adjoint.

Abstraction Type

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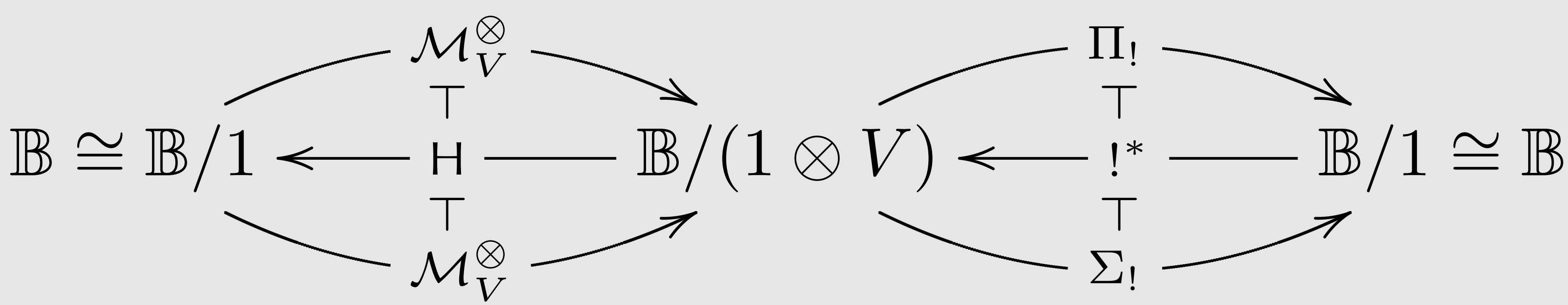
Proof:

$$\mathbb{B} \xrightarrow[-\otimes V]{} \mathbb{B} \quad = \quad \mathbb{B} \cong \mathbb{B}/1 \xrightarrow[\mathcal{M}_V^{\otimes}]{} \mathbb{B}/(1 \otimes V) \xrightarrow[\Sigma_!]{} \mathbb{B}/1 \cong \mathbb{B}$$



Abstraction Type – Properties

In a category with binding structure that has finite sums, finite limits and is cartesian closed, there are natural isomorphisms:

$$\begin{aligned}
V \multimap 0 &\cong 0 \\
V \multimap 1 &\cong 1 \\
V \multimap (A + B) &\cong (V \multimap A) + (V \multimap B) \\
V \multimap (A \times B) &\cong (V \multimap A) \times (V \multimap B) \\
V \multimap (A \Rightarrow B) &\cong (V \multimap A) \Rightarrow (V \multimap B)
\end{aligned}$$

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$$V \multimap (A + B) \cong (V \multimap A) + (V \multimap B)$$

$$V \multimap (A$$

$$V \multimap (A \Rightarrow$$

Proof: H is part of an equivalence and therefore preserves all categorical constructions.

Abstraction Type – Nominal Sets

- In Nom, $\mathbb{A} \multimap -$ is the abstraction set

$$\begin{aligned} \mathbb{A} \multimap X &= \{a.x \mid a \in \mathbb{A} \ \wedge \ x \in X\} \\ &= [\mathbb{A}]X \end{aligned}$$

- Counit = Concretion

$$\begin{aligned} \varepsilon_X &: (\mathbb{A} \multimap X) * \mathbb{A} \longrightarrow X \\ \varepsilon_X &: \quad \langle y, a \rangle \quad \longmapsto \quad y@a \end{aligned}$$

Abstraction Type – Operations

- Operations on abstraction sets are captured by the (co)units of the adjunctions $\mathsf{H} \dashv \mathcal{M}_V^{\otimes} \dashv \mathsf{H}$

$$\begin{array}{rcl} \text{Concretion} & — & \text{Counit of } \mathcal{M}_V^{\otimes} \dashv \mathsf{H} \\ \text{Binding} & — & \text{Unit of } \mathsf{H} \dashv \mathcal{M}_V^{\otimes} \end{array}$$

- Binding is the inverse of concretion
- Equations such as $(a.y)@a = y$ follow from the equations for the units and counits

$$\eta^{-1} = \varepsilon'$$
$$\varepsilon^{-1} = \eta'$$

Abstraction Type – Binding

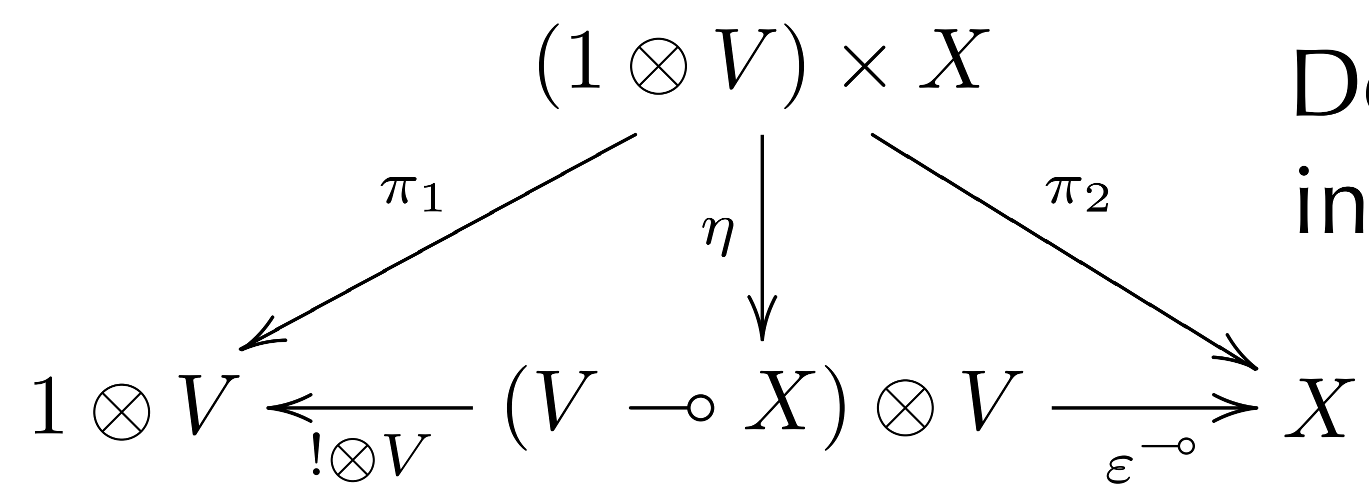
$$
\begin{array}{ccccc}
X & \xrightarrow{\eta_f} & (\mathrm{H}X)\otimes V & \xrightarrow{\varepsilon'_f} & X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (\mathrm{H}f)\otimes V} & & \downarrow{\scriptstyle f} \\
I\otimes V & = & I\otimes V & = & I\otimes V
\end{array}
$$

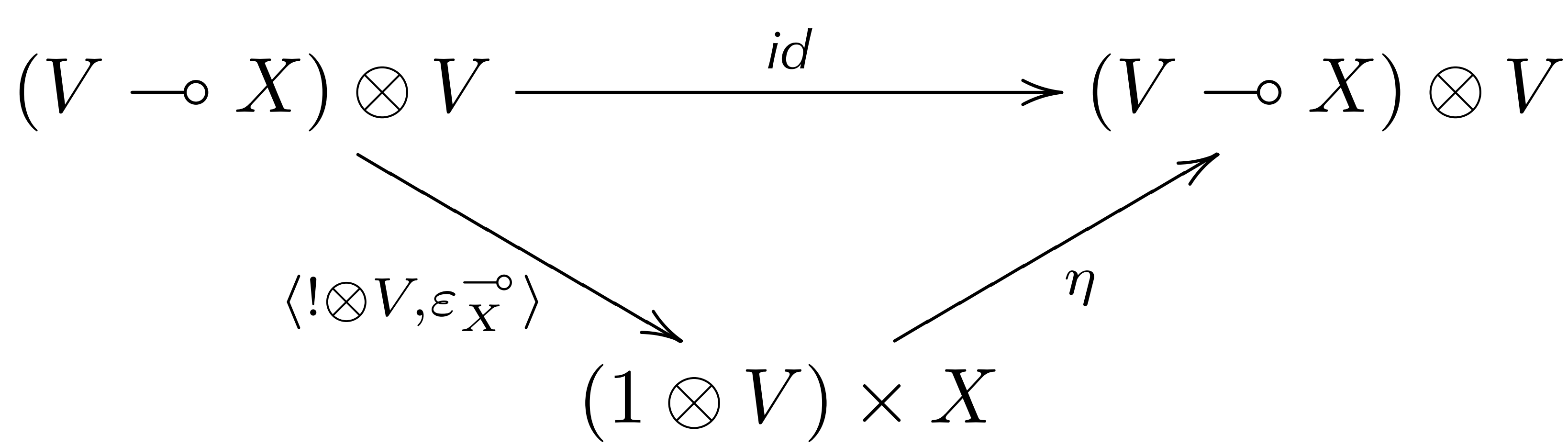
$$
\begin{array}{ccccc}
(1\otimes V)\times X & \xrightarrow{\eta} & (V \multimap X)\otimes V & \xrightarrow{\varepsilon'} & (1\otimes V)\times X \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle !\otimes V} & & \downarrow{\scriptstyle \pi_1} \\
1\otimes V & = & 1\otimes V & = & 1\otimes V.
\end{array}
$$

Abstraction Type – Binding

In any category with binding structure the diagrams below commute.

Def. of Binding in [Menni 2003]



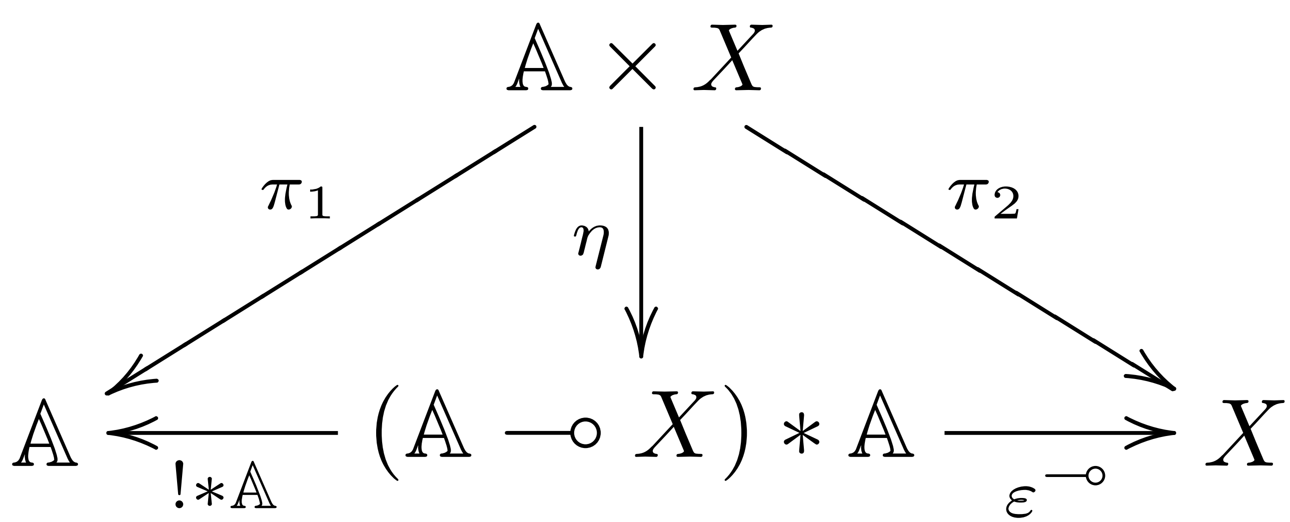


Nominal Sets

- The unit η contains the binding operation $a.x$.

$$\eta(a, x) = \langle a.x, a \rangle$$

- Commuting diagrams contain the information that a is fresh for $a.x$ and explain the interaction of binding and concretion.



$$a\#(a.x)$$

$$(a.x)@a = x$$

Nominal Sets

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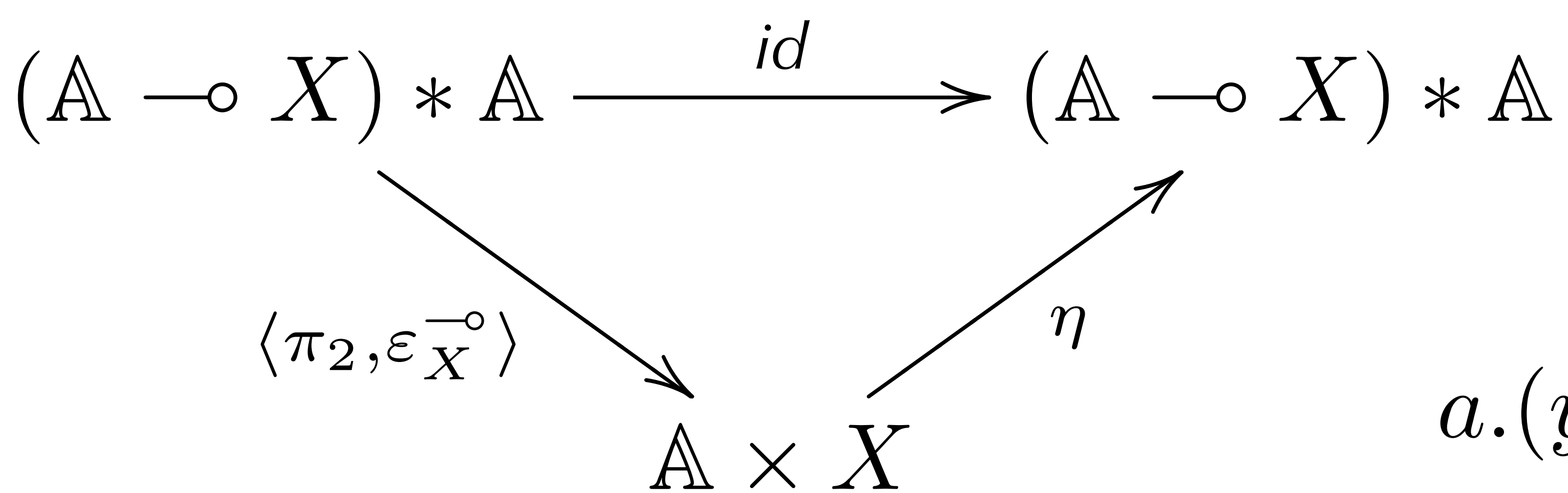

$$a.(y@a) = y$$

H – Abstraction Type

Binding

Unit η of H $\dashv \mathcal{M}_V^{\otimes}$ $\quad \langle a, x \rangle \longmapsto a.x \# a$

Concretion

Counit ε of $\mathcal{M}_V^{\otimes} \dashv$ H $\quad y \# a \longmapsto y@a$

Equations

$\eta' = \varepsilon^{-1}$ and $\varepsilon' = \eta^{-1}$

$$(a.x)@a = x$$

$$a.(y@a) = y$$

$$\ldots$$

Some/Any Quantifier

In the subobject logic of any category with binding structure, there is a some/any quantifier.

$$\frac{\Gamma\#\, v\colon V \mid \varphi\#v \vdash \psi}{\Gamma \mid \varphi \vdash \mathrm{H}v{:}V.\,\psi} \qquad \frac{\Gamma\#\, v\colon V \mid \varphi \vdash \psi\#v}{\Gamma \mid \mathrm{H}v{:}V.\,\varphi \vdash \psi}$$

Both ∇ and И can be modelled by this quantifier (in different binding structures).

Some/Any Quantifier

The quantifier H has very nice properties.

$$
\begin{aligned}
\mathsf{H}x.\,\bot &\iff \bot \\
\mathsf{H}x.\,\top &\iff \top \\
\mathsf{H}x.\,\varphi \vee \psi &\iff (\mathsf{H}x.\,\varphi) \vee (\mathsf{H}x.\,\psi) \\
\mathsf{H}x.\,\varphi \wedge \psi &\iff (\mathsf{H}x.\,\varphi) \wedge (\mathsf{H}x.\,\psi) \\
\mathsf{H}x.\,\varphi \supset \psi &\iff (\mathsf{H}x.\,\varphi) \supset (\mathsf{H}x.\,\psi)
\end{aligned}
$$

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Simple Monoidal Products and Sums

- Simple products and sums ($\pi_X^* \cong \mathcal{M}_X^{\times}$)

$$\Sigma_X \dashv \mathcal{M}_X^{\times} \dashv \Pi_X$$

- Simple monoidal products and sums

$$\Sigma_X^{\otimes} \dashv \mathcal{M}_X^{\otimes} \dashv \Pi_X^{\otimes}$$

- $\mathrm{H} \dashv \mathcal{M}_V^{\otimes} \dashv \mathrm{H}$ implies $\Sigma_V^{\otimes} \cong \mathrm{H} \cong \Pi_V^{\otimes}$

Simple Monoidal Products and Sums

Adjunction $\mathcal{M}_X^{\otimes} \dashv \Pi_X^{\otimes}$

$$\frac{\Gamma \otimes x{:}X \vdash M : Y}{\Gamma \vdash \lambda^{\otimes} x{:}X.\, M \;:\; \Pi^{\otimes} x{:}X.\, Y}$$

$$\frac{\Gamma \vdash M : \Pi^{\otimes} x{:}X.\, Y \qquad \Delta \vdash N : X}{\Gamma \otimes \Delta \vdash M @ N \;:\; Y[N/x]}$$

Adjunction $\Sigma_X^{\otimes} \dashv \mathcal{M}_X^{\otimes}$

$$\frac{\Gamma \vdash M : X \qquad \Delta \vdash N : Y}{\Gamma \otimes \Delta \vdash M.N : (\Sigma^{\otimes} x{:}X.\, Y)^{\otimes M}}$$

$$\frac{(\Gamma \otimes x{:}X),\, y{:}Y \vdash N : Z^{\otimes x}}{\Gamma,\; z{:}(\Sigma^{\otimes} x{:}X.\, Y) \vdash \text{let } z \text{ be } x.y \text{ in } N : Z}$$

H as a Sum/Product Type

- H is a sum/product type ($\Sigma_V^{\otimes} \cong \mathsf{H} \cong \Pi_V^{\otimes}$)
- Operations and equations from $\Pi_V^{\otimes}$ and $\Sigma_V^{\otimes}$
- Equations from $\eta^{-1} = \varepsilon'$ and $\varepsilon^{-1} = \eta'$:

$$(x.M)@x = M \qquad \qquad \dots$$
$$x.(N@x) = N$$

These equations explain the interaction of the two views of H as $\Pi_V^{\otimes}$ and $\Sigma_V^{\otimes}$.

Categories with Binding Structure

- Universal characterisation of name-binding
- Instances
  - Nominal Sets
  - Nominal Realisability
  - Species of Structures ($\mathrm{Sets}^{\mathbf{B}}$)
  - Menni's axiomatisation
- Many other presheaf-categories do not enjoy all of binding structure (e.g. $\mathrm{Sets}^{\mathbf{F}}$, $\mathrm{Sets}^{\mathbf{I}}$)
- ***No*** assumptions on $\otimes$ or V

Categories with Bindable Names

- Add axioms to make $\otimes$ like a freshness relation, e.g. that the following map is an isomorphism.

$$[\delta, \imath] \colon V + (V \otimes V) \longrightarrow V \times V$$

- Consequences
  - Object V is infinite
  - Rules for quantifier H are like those for И
  - In boolean toposes equivalent to Menni's axioms

Conclusion and Further Work

Binding structure is a simple, direct, universal characterisation of name binding.

Conjectures

- $\mathbf{Sets}^{\mathbf{B}}$ is obtained by freely adding *binding structure* to Sets.
- Nom is obtained by freely adding *bindable names* to Sets.