Modelling Generic Judgements

How does ∇ relate to И?

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Part I
A Simple Semantics for ∇

Classical FOλ$^\nabla$

First-order classical logic over a simply-typed λ-calculus

- ∇ restricted to *nominal types* $\nabla x{:}\,\iota.\varphi$
- Classical version of sequent calculus with generic judgements [Miller & Tiu '03]

$$\Gamma \mid \sigma_1 \rhd \varphi_1,\ \ldots,\ \sigma_n \rhd \varphi_n \vdash \sigma'_1 \rhd \psi_1,\ \ldots,\ \sigma'_m \rhd \psi_m$$

Local contexts of nominal types

Interpreting Types and Terms

Simply-typed λ-calculus, Higher Order Abstract Syntax

$$\begin{array}{lcl} \text{app} & : & Tm \times Tm \to Tm \\ \text{lam} & : & (Tm \Rightarrow Tm) \to Tm \end{array}$$

Object-level term with n free variables

$$\underbrace{Tm \times \ldots \times Tm}_{n \text{ times}} \longrightarrow Tm$$

Model should not be all syntax!

$\Rightarrow$ HOAS adequate only for nominal types

Interpreting Types and Terms

Sets indexed by local contexts

[Hofmann '99], [Fiore, Plotkin, Turi '99],...

Object A

$A\sigma$ — Family of sets

$(-)[\theta]\colon\ A\sigma \to A\sigma'$ — Substitution action

$$[\![\iota]\!]\sigma = \{M \mid \sigma \vdash M : \iota\}$$

Morphism $M\colon\ \Gamma \to A$

$M_\sigma\colon\ \Gamma\sigma \to A\sigma$ — Family of functions

$$M_\sigma(x)[\theta] = M_{\sigma'}(x[\theta])$$

Interpreting Types and Terms

Sets indexed by local contexts

[Hofmann '99], [Fiore, Plotkin, Turi '99],...

Object A

$A\sigma$

$(-)[\theta]\colon\ A\sigma \to A\sigma'$

$M\colon [\![\iota]\!] \to [\![\iota]\!]$ determined by

$M_{x:\iota}(x) \in [\![\iota]\!]x{:}\iota$

$$[\![\iota]\!]\sigma = \{M \mid \sigma \vdash M : \iota\}$$

Morphism $M\colon\ \Gamma \to A$

$M_\sigma\colon\ \Gamma\sigma \to A\sigma$ — Family of functions

$$M_\sigma(x)[\theta] = M_{\sigma'}(x[\theta])$$

Interpreting Predicates

Interpretation of a formula $\Gamma \vdash \varphi$

Obvious idea:

$$\varphi\sigma \subseteq \Gamma\sigma$$

$$x \in \varphi\sigma \implies x[\theta] \in \varphi\sigma'$$

Problem:

$x : \iota,\ y : \iota \vdash x \neq y$ and $\theta = [x/x, x/y]$

[Hofmann '99]

Interpreting Predicates

Interpretation of a formula $\Gamma \vdash \varphi$

Instead:

$$\varphi\sigma \subseteq \Gamma\sigma$$

$$x \in \varphi\sigma \implies x[\alpha] \in \varphi\sigma'$$

where α is an order-preserving renaming:

$$[x_1/y_1]\ldots[x_n/y_n] \colon (x_1 : \iota_1, \ldots, x_n : \iota_n) \longrightarrow (y_1 : \iota_1, \ldots, y_n : \iota_n)$$

Interpreting Predicates

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Why not injections? $\nabla x : \iota. \varphi \not\supset \forall x : \iota. \varphi, \ldots$

Satisfaction

$$
\begin{aligned}
\sigma \Vdash_{\rho,\Gamma} R(t) &\iff t[\rho] \in [\![R]\!]\sigma \\
\sigma \Vdash_{\rho,\Gamma} \top &\iff \text{always} \\
\sigma \Vdash_{\rho,\Gamma} \varphi \vee \psi &\iff \sigma \Vdash_{\rho,\Gamma} \varphi \text{ or } \sigma \Vdash_{\rho,\Gamma} \psi \\
\sigma \Vdash_{\rho,\Gamma} \varphi \wedge \psi &\iff \sigma \Vdash_{\rho,\Gamma} \varphi \text{ and } \sigma \Vdash_{\rho,\Gamma} \psi \\
\sigma \Vdash_{\rho,\Gamma} \varphi \supset \psi &\iff \sigma \Vdash_{\rho,\Gamma} \varphi \text{ implies } \sigma \Vdash_{\rho,\Gamma} \psi \\
\sigma \Vdash_{\rho,\Gamma} \exists x{:}\tau.\, \varphi &\iff \sigma \Vdash_{\rho[e/x],(\Gamma,\, x:\tau)} \varphi \text{ for some } e \in [\![\tau]\!]\sigma \\
\sigma \Vdash_{\rho,\Gamma} \forall x{:}\tau.\, \varphi &\iff \sigma \Vdash_{\rho[e/x],(\Gamma,\, x:\tau)} \varphi \text{ for all } e \in [\![\tau]\!]\sigma \\
\sigma \Vdash_{\rho,\Gamma} \nabla x{:}\iota.\, \varphi &\iff \sigma,\ x{:}\iota \Vdash_{\rho[x/x],(\Gamma,\, x:\iota)} \varphi \\
&\qquad \text{for some/any fresh } \textit{variable } x
\end{aligned}
$$

Formula φ is valid if $\cdot \Vdash_{\rho,\Gamma} \varphi$ holds for all ρ.

Generic Judgements and Raising

Translate $\sigma \rhd \varphi$ to $\nabla \sigma.\, \varphi$

$$\frac{\Gamma,\, h\colon \sigma \to \tau \mid \Phi,\, \sigma \rhd \varphi[h\,\sigma/x] \vdash \Psi}{\Gamma \mid \Phi,\, \sigma \rhd \exists x{:}\tau.\, \varphi \vdash \Psi}$$

Generic Judgements and Raising

Translate $\sigma \rhd \varphi$ to $\nabla\sigma.\,\varphi$

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Generic Judgements and Raising

Translate $\sigma \triangleright \varphi$ to $\nabla \sigma.\, \varphi$

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$$\sigma \Vdash_{\rho,\Gamma} \nabla x\colon \iota.\, \exists y\colon \tau.\, \varphi$$

iff

$$\sigma \Vdash_{\rho,\Gamma} \exists h\colon \iota \to \tau.\, \nabla x\colon \iota.\, \varphi[h\, x / y]$$

Generic Judgements and Raising

Translate $\sigma \rhd \varphi$ to $\nabla \sigma.\, \varphi$

$$\frac{\Gamma,\, h\colon \sigma \to \tau \mid \Phi,\, \nabla \sigma.\, \varphi[h\,\sigma/x] \vdash \Psi}{\Gamma \mid \Phi,\, \exists h{:}\sigma \to \tau.\, \nabla \sigma.\, \varphi[h\,\sigma/x] \vdash \Psi}$$

$$\sigma \Vdash_{\rho,\Gamma} \nabla x\colon \iota.\, \exists y\colon \tau.\, \varphi$$

iff

$$\sigma \Vdash_{\rho,\Gamma} \exists h\colon \iota \to \tau.\, \nabla x\colon \iota.\, \varphi[h\,x/y]$$

Soundness & Completeness

Soundness

Any derivable sequent is valid.

Completeness

Any valid sequent $\cdot \mid \Phi \vdash \Psi$ is derivable.

Part II
Binding Structure, ∇ and $\reflectbox{N}$

Binding Structure

Working with α-equivalence classes is the same as working with freshly named instances.

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A *category with binding structure* is a triple $(\mathbb{B}, \otimes, V)$...

Category with finite limits

Monoidal Structure (freshness)

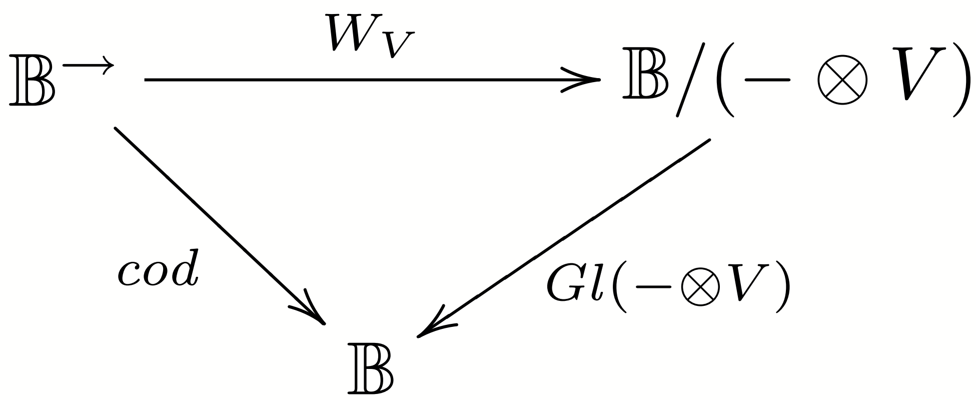
Object (names)

$$\begin{aligned} \mathbb{B} &= \text{FM sets} \\ V &= \mathbb{A} \\ A \otimes B &= \{\langle a, b\rangle \in A \times B \mid a\#b\} \end{aligned}$$

Binding Structure

Working with α-equivalence classes is the same as working with freshly named instances.

A *category with binding structure* is a triple $(\mathbb{B}, \otimes, V)$ such that the functor $W_V : f \mapsto f \otimes V$ is an equivalence of fibrations.



Codomain Fibration

Working with α-equivalence classes is the same as working with freshly named instances.

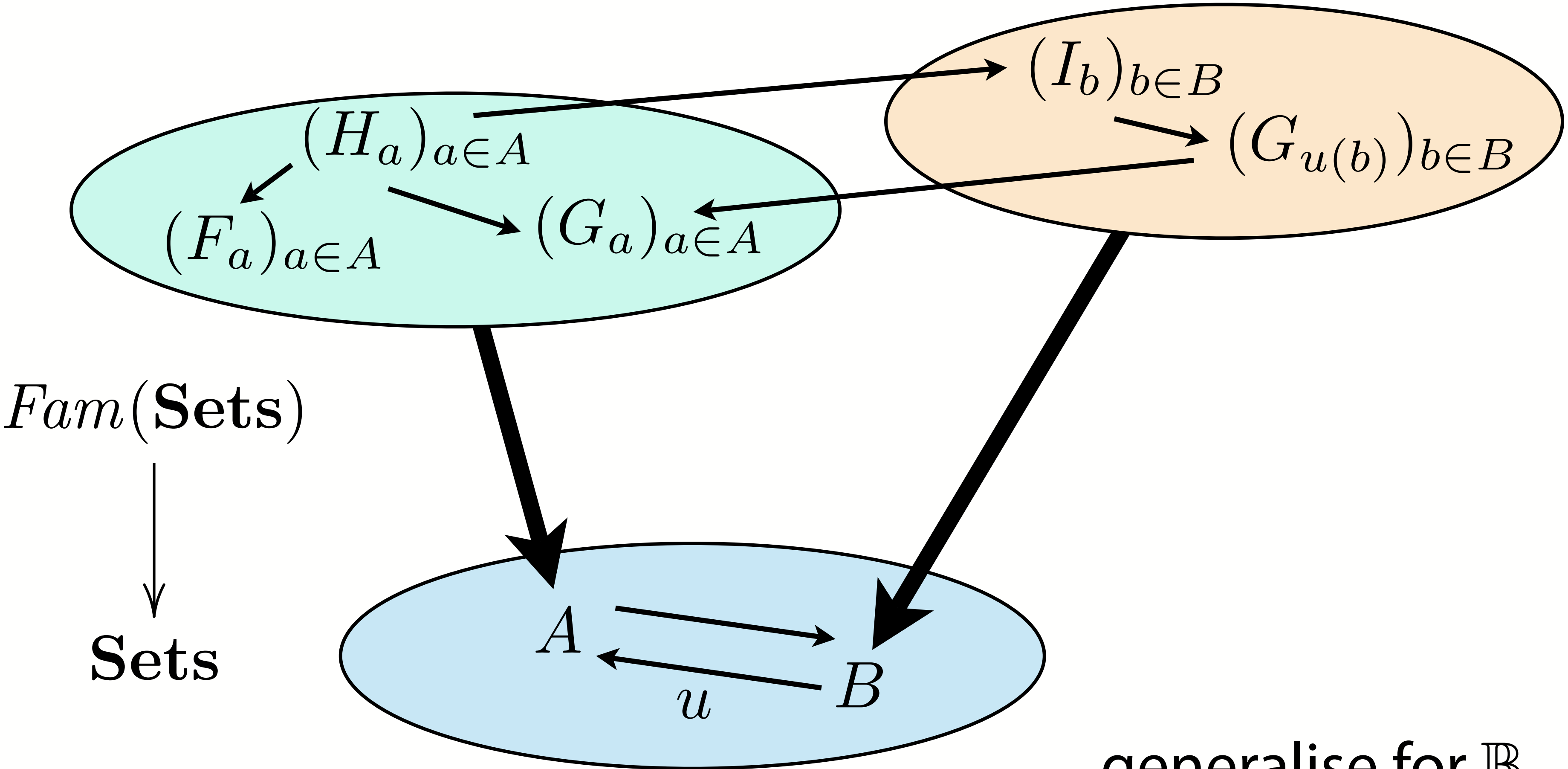
$$\begin{array}{c} \mathbb{B}^{\rightarrow} \\ \downarrow{\scriptstyle cod} \\ \mathbb{B} \end{array}$$

- Category with pullbacks $\mathbb{B}$
- Families of $\mathbb{B}$-objects indexed by $\mathbb{B}$-objects
- Dependent type theory $\Gamma \vdash A$

Other choices possible, e.g. subobject-logic

Codomain Fibration

Families of sets



...generalise for $\mathbb{B}$

Codomain Fibration

$$\begin{array}{c} Fam(\mathbf{Sets}) \\ \downarrow \\ \mathbf{Sets} \end{array} \quad \text{is equivalent to} \quad \begin{array}{c} \mathbf{Sets}^{\rightarrow} \\ \downarrow{\scriptstyle cod} \\ \mathbf{Sets} \end{array}$$

$$(B_a)_{a\in A} \longmapsto \begin{array}{c} \sum_{a\in A} B_a \\ {\scriptstyle \pi_1}\downarrow \\ A \end{array} \qquad \begin{array}{c} B \\ {\scriptstyle f}\downarrow \\ A \end{array} \longmapsto (f^{-1}(a))_{a\in A}$$

Codomain Fibration

$$
\begin{array}{ccc}
\mathbb{B}^{\to} & & \begin{array}{ccc} F & \xrightarrow{v} & G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[u]{} & B \end{array} \\
{\scriptstyle cod}\downarrow & & \\
\mathbb{B} & & A \xrightarrow[u]{} B
\end{array}
$$

In $\mathbf{Sets}$:

$$\big(v_a \colon F_a \to G_{u(a)}\big)_{a \in A}$$

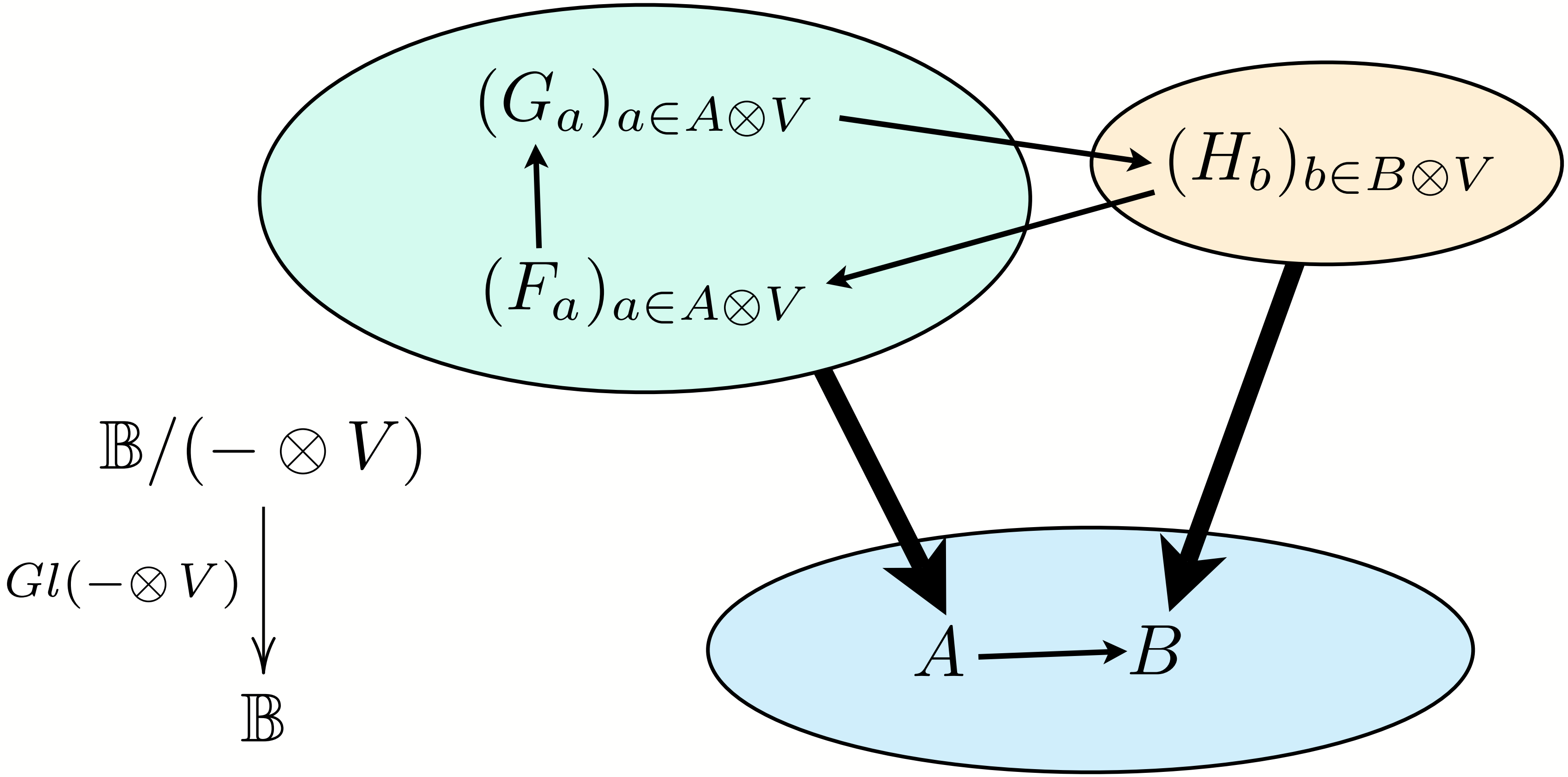
Glueing a Fresh Name

Working with α-equivalence classes is the same as
working with freshly named instances.

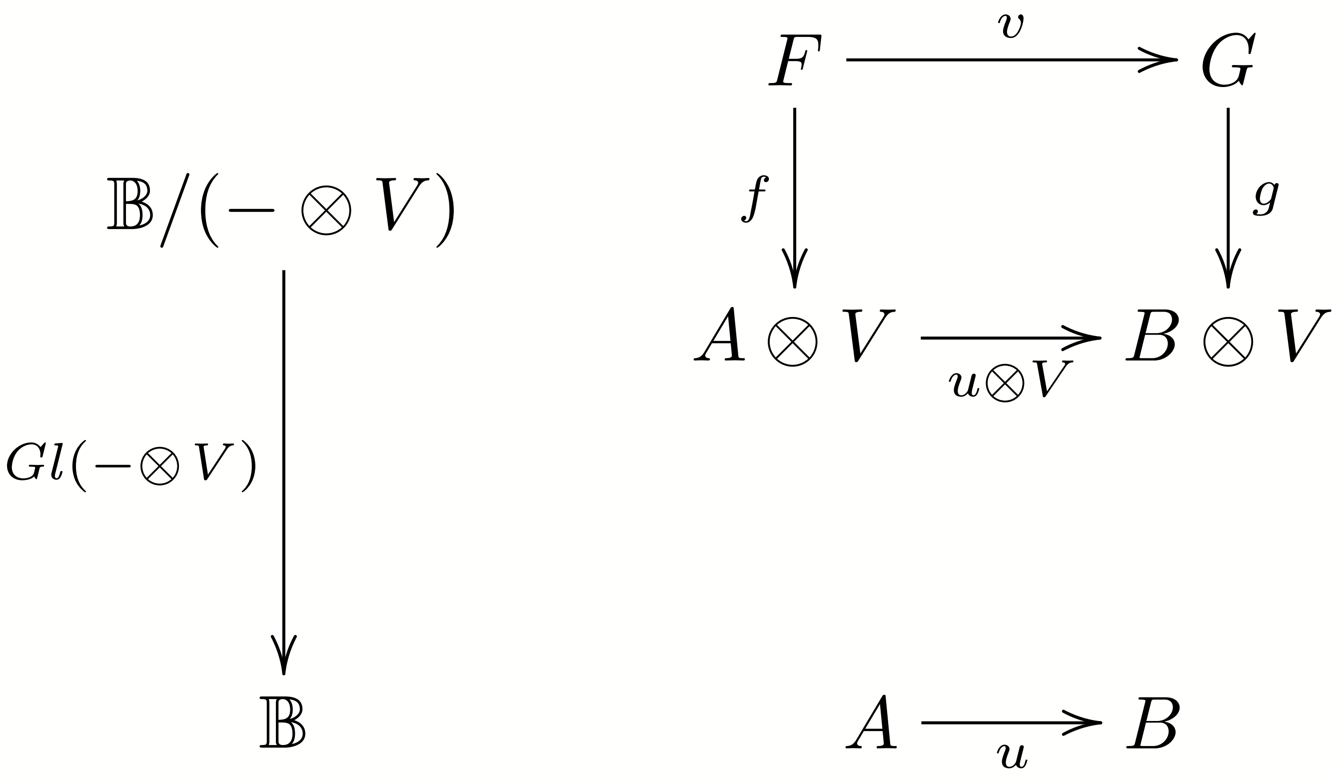
$$\begin{array}{c} \mathbb{B}/(- \otimes V) \\ \Big\downarrow {\scriptstyle Gl(-\otimes V)} \\ \mathbb{B} \end{array}$$

- Families of $\mathbb{B}$-objects indexed by $\mathbb{B}$-objects of the form $A \otimes V$
- Dependent type theory with judgements $\Gamma \# v{:}V \vdash A$

Glueing a Fresh Name



Glueing a Fresh Name



Adding a Fresh Name

$$
\begin{array}{ccc}
\mathbb{B}^{\rightarrow} & \xrightarrow{W_V} & \mathbb{B}/(-\otimes V) \\
& {\scriptstyle cod} \searrow \quad \swarrow {\scriptstyle Gl(-\otimes V)} & \\
& \mathbb{B} &
\end{array}
$$

Adding a Fresh Name

$$\begin{array}{c} B \\ {\scriptstyle f}\big\downarrow \\ A \end{array} \qquad \longmapsto \qquad \begin{array}{c} B \otimes V \\ {\scriptstyle f \otimes V}\big\downarrow \\ A \otimes V \end{array}$$

$$\begin{array}{ccc} \mathbb{B}^{\rightarrow} & \xrightarrow{\;W_V\;} & \mathbb{B}/(- \otimes V) \\ & {\scriptstyle cod}\searrow \quad \swarrow{\scriptstyle Gl(-\otimes V)} & \\ & \mathbb{B} & \end{array}$$

Adding a Fresh Name

$$(F_a)_{a \in A} \qquad \longmapsto \qquad \big(F_a \otimes \{v\}\big)_{\langle a, v\rangle \in A \otimes V}$$

$$\begin{array}{ccc}
\mathbb{B}^{\to} & \xrightarrow{\;W_V\;} & \mathbb{B}/(- \otimes V) \\
 & {\scriptstyle cod} \searrow \quad \swarrow {\scriptstyle Gl(-\otimes V)} & \\
 & \mathbb{B} &
\end{array}$$

Binding Structure

Working with α-equivalence classes is the same as working with freshly named instances.

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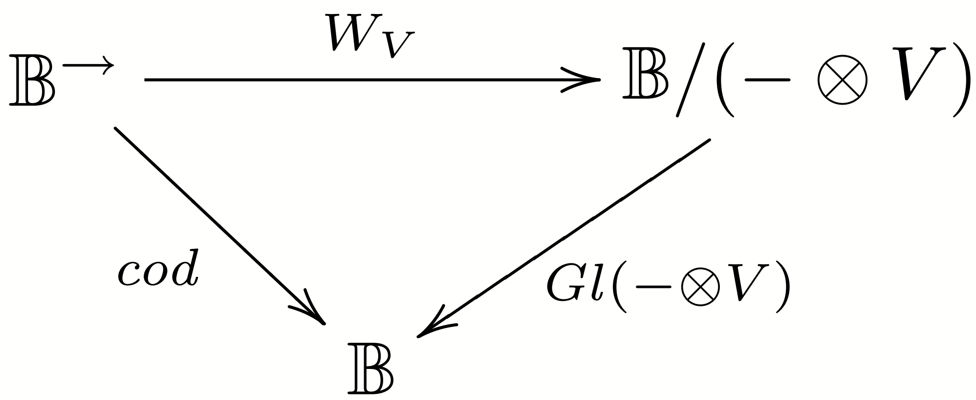
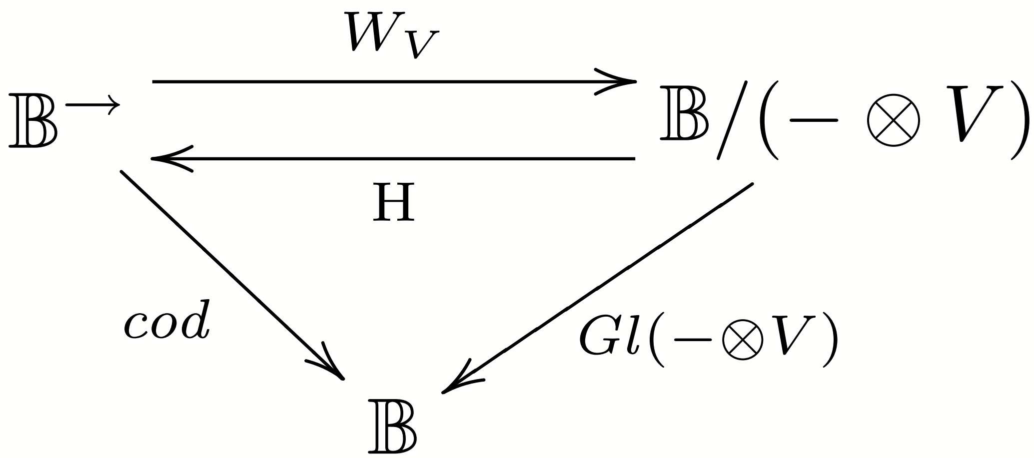


The Equivalence



Natural vertical Isomorphisms

$$W_V \circ \mathrm{H} \cong Id$$

$$\mathrm{H} \circ W_V \cong Id$$

W_V — Add a fresh name

H — Bind a name (take α-equivalence class)

FM sets:

$$W_V(F_a)_{a\in A} = (F_a \otimes \{v\})_{\langle a,v\rangle \in A\otimes V}$$

$$\mathrm{H}(G_{\langle a,v\rangle})_{\langle a,v\rangle \in A\otimes V} = ([v\in V]G_{\langle a,v\rangle})_{a\in A}$$

Binding Structure implies...

... FM-like constructs N, $[\mathbb{A}]X,\ n.x,\ x@n,\ new\ n.x$

[Gabbay & Pitts]

$\mathrm{H} \dashv W_V \dashv \mathrm{H}$ with $\eta' = \varepsilon^{-1}$ and $\varepsilon' = \eta^{-1}$

Binding Structure implies...

... FM-like constructs И, $[\mathbb{A}]X,\ n.x,\ x@n,\ new\ n.x$
[Gabbay & Pitts]

$\mathrm{H} \dashv W_V \dashv \mathrm{H}$ with $\eta' = \varepsilon^{-1}$ and $\varepsilon' = \eta^{-1}$

$$
\begin{aligned}
\mathrm{H}0 &\cong 0 \\
\mathrm{H}1 &\cong 1 \\
\mathrm{H}(A+B) &\cong \mathrm{H}A + \mathrm{H}B \\
\mathrm{H}(A\times B) &\cong \mathrm{H}A \times \mathrm{H}B \\
\mathrm{H}(A\Rightarrow B) &\cong \mathrm{H}A \Rightarrow \mathrm{H}B
\end{aligned}
$$

H — New-Name Quantifier

H restricts to a some/any quantifier

$$\frac{\Gamma \# v : V \mid \varphi \# v \vdash \psi}{\Gamma \mid \varphi \vdash \mathrm{H} v{:}V.\, \psi} \qquad \frac{\Gamma \# v : V \mid \varphi \vdash \psi \# v}{\Gamma \mid \mathrm{H} v{:}V.\, \varphi \vdash \psi}$$

Under certain circumstances:

$$\frac{\Gamma \# v : V \mid \varphi \vdash \psi}{\Gamma \mid \varphi \vdash \mathrm{H} v{:}V.\, \psi} \qquad \frac{\Gamma \# v : V \mid \varphi \vdash \psi}{\Gamma \mid \mathrm{H} v{:}V.\, \varphi \vdash \psi}$$

— the rules for И!

H — Abstraction Set

Binding

Unit η of $\mathrm{H} \dashv W_V$ $\qquad \langle v, x \rangle \longmapsto v.x \# v$

Concretion

Counit ε of $W_V \dashv \mathrm{H}$ $\qquad y \# v \longmapsto y@v$

Equations

$\eta' = \varepsilon^{-1}$ and $\varepsilon' = \eta^{-1}$

$$(v.x)@v = x$$

$$v.(y@v) = y$$

Binding Structure and FOλ$^{\nabla}$

Interpretation of Types

As in Part I

$A\sigma$	— Family of sets
$(-)[\theta]\colon\ A\sigma \to A\sigma'$	— Substitution action

Interpretation of the Logic

Internal logic of a category with binding structure

∇	— New-quantifier И
Raising	— Binding for И

Binding Structure and ∇

Binding Structure	$FO\lambda^{\nabla}$-Interpretation
Linear Species	Part I
Species	Part I $+ \nabla x.\nabla y.\varphi \Leftrightarrow \nabla y.\nabla x.\varphi$
FM Sets	[Gabbay & Cheney '04] $\nabla x.\varphi = \text{И} x.\varphi[\mathsf{n}(x)/x]$
?	[Miculan & Yemane '05]

Conclusion

∇ and И are both H, in different binding structures

Further Work

- Simple Semantics for Intuitionistic $FO\lambda^{\nabla}$
- Proof theory of $FO\lambda^{\nabla}$ for Nominal Logic
- Relation to [Miculan & Yemane '05]
- a-Logic [Gabbay & Gabbay '05]