

# Modelling Generic Judgements

How does  $\nabla$  relate to  $\mathbb{I}$ ?

Ulrich Schöpp  
LMU Munich

# Part I

## A Simple Semantics for $\nabla$

# Classical FO $\lambda^\nabla$

First-order classical logic over a simply-typed  $\lambda$ -calculus

- $\nabla$  restricted to *nominal types*  $\nabla x: \iota. \varphi$
- Classical version of sequent calculus with generic judgements [Miller & Tiu '03]

$$\Gamma \mid \sigma_1 \triangleright \varphi_1, \dots, \sigma_n \triangleright \varphi_n \vdash \sigma'_1 \triangleright \psi_1, \dots, \sigma'_m \triangleright \psi_m$$

Local contexts of nominal types



# Interpreting Types and Terms

Simply-typed  $\lambda$ -calculus, Higher Order Abstract Syntax

$$\begin{array}{lcl} \text{app} & : & Tm \times Tm \rightarrow Tm \\ \text{lam} & : & (Tm \Rightarrow Tm) \rightarrow Tm \end{array}$$

Object-level term  
with  $n$  free variables

$$\underbrace{Tm \times \dots \times Tm}_{n \text{ times}} \longrightarrow Tm$$

Model should not be all syntax!

$\Rightarrow$  HOAS adequate only for nominal types

# Interpreting Types and Terms

Sets indexed by local contexts

[Hofmann '99], [Fiore, Plotkin, Turi '99],...

Object  $A$

$A_\sigma$  — Family of sets

$(-)[\theta]: A_\sigma \rightarrow A_{\sigma'}$  — Substitution action

$$[[\iota]]_\sigma = \{M \mid \sigma \vdash M : \iota\}$$

Morphism  $M: \Gamma \rightarrow A$

$M_\sigma: \Gamma_\sigma \rightarrow A_\sigma$  — Family of functions

$$M_\sigma(x)[\theta] = M_{\sigma'}(x[\theta])$$

# Interpreting Types and Terms

Sets indexed by local contexts

[Hofmann '99], [Fiore, Plotkin, Turi '99],...

Object  $A$

$A_\sigma$

$(-)[\theta]: A_\sigma \rightarrow A_{\sigma'}$

$$[[\iota]]_\sigma = \{M \mid \sigma \vdash M : \iota\}$$

$M: [[\iota]] \rightarrow [[\iota]]$  determined by

$$M_{x:\iota}(x) \in [[\iota]]_{x:\iota}$$

Morphism  $M: \Gamma \rightarrow A$

$M_\sigma: \Gamma_\sigma \rightarrow A_\sigma$  — Family of functions

$$M_\sigma(x)[\theta] = M_{\sigma'}(x[\theta])$$

# Interpreting Predicates

Interpretation of a formula  $\Gamma \vdash \varphi$

Obvious idea:

$$\varphi\sigma \subseteq \Gamma\sigma$$
$$x \in \varphi\sigma \implies x[\theta] \in \varphi\sigma'$$

Problem:

$$x : \iota, y : \iota \vdash x \neq y \text{ and } \theta = [x/x, x/y]$$

[Hofmann '99]

# Interpreting Predicates

Interpretation of a formula  $\Gamma \vdash \varphi$

Instead:

$$\varphi\sigma \subseteq \Gamma\sigma$$
$$x \in \varphi\sigma \implies x[\alpha] \in \varphi\sigma'$$

where  $\alpha$  is an order-preserving renaming:

$$[x_1/y_1] \dots [x_n/y_n]: (x_1 : \iota_1, \dots, x_n : \iota_n) \longrightarrow (y_1 : \iota_1, \dots, y_n : \iota_n)$$

# Interpreting Predicates

Interpretation of a formula  $\Gamma \vdash \varphi$

Instead:

$$\begin{aligned} \varphi\sigma &\subseteq \Gamma\sigma \\ x \in \varphi\sigma &\implies x[\alpha] \in \varphi\sigma' \end{aligned}$$

where  $\alpha$  is an order-preserving renaming:

$$[x_1/y_1] \dots [x_n/y_n]: (x_1: \iota_1, \dots, x_n: \iota_n) \longrightarrow (y_1: \iota_1, \dots, y_n: \iota_n)$$

Why not injections?  $\nabla x: \iota. \varphi \not\equiv \forall x: \iota. \varphi, \dots$

# Satisfaction

$\sigma \Vdash_{\rho, \Gamma} R(t)$	$\iff$	$t[\rho] \in \llbracket R \rrbracket \sigma$
$\sigma \Vdash_{\rho, \Gamma} \top$	$\iff$	<b>always</b>
$\sigma \Vdash_{\rho, \Gamma} \varphi \vee \psi$	$\iff$	$\sigma \Vdash_{\rho, \Gamma} \varphi$ <b>or</b> $\sigma \Vdash_{\rho, \Gamma} \psi$
$\sigma \Vdash_{\rho, \Gamma} \varphi \wedge \psi$	$\iff$	$\sigma \Vdash_{\rho, \Gamma} \varphi$ <b>and</b> $\sigma \Vdash_{\rho, \Gamma} \psi$
$\sigma \Vdash_{\rho, \Gamma} \varphi \supset \psi$	$\iff$	$\sigma \Vdash_{\rho, \Gamma} \varphi$ <b>implies</b> $\sigma \Vdash_{\rho, \Gamma} \psi$
$\sigma \Vdash_{\rho, \Gamma} \exists x:\tau. \varphi$	$\iff$	$\sigma \Vdash_{\rho[e/x], (\Gamma, x:\tau)} \varphi$ <b>for some</b> $e \in \llbracket \tau \rrbracket \sigma$
$\sigma \Vdash_{\rho, \Gamma} \forall x:\tau. \varphi$	$\iff$	$\sigma \Vdash_{\rho[e/x], (\Gamma, x:\tau)} \varphi$ <b>for all</b> $e \in \llbracket \tau \rrbracket \sigma$
$\sigma \Vdash_{\rho, \Gamma} \nabla x:\iota. \varphi$	$\iff$	$\sigma, x:\iota \Vdash_{\rho[x/x], (\Gamma, x:\iota)} \varphi$ <b>for some/any fresh variable <math>x</math></b>

Formula  $\varphi$  is valid if  $\cdot \Vdash_{\rho, \Gamma} \varphi$  holds for all  $\rho$ .

# Generic Judgements and Raising

Translate  $\sigma \triangleright \varphi$  to  $\nabla \sigma. \varphi$

$$\frac{\Gamma, h: \sigma \rightarrow \tau \mid \Phi, \sigma \triangleright \varphi[h \sigma/x] \vdash \Psi}{\Gamma \mid \Phi, \sigma \triangleright \exists x:\tau. \varphi \vdash \Psi}$$

# Generic Judgements and Raising

Translate  $\sigma \triangleright \varphi$  to  $\nabla\sigma.\varphi$

$$\frac{\Gamma, h: \sigma \rightarrow \tau \mid \Phi, \nabla\sigma.\varphi[h\sigma/x] \vdash \Psi}{\Gamma \mid \Phi, \nabla\sigma.\exists x:\tau.\varphi \vdash \Psi}$$

# Generic Judgements and Raising

Translate  $\sigma \triangleright \varphi$  to  $\nabla \sigma. \varphi$

$$\frac{\Gamma, h: \sigma \rightarrow \tau \mid \Phi, \nabla \sigma. \varphi[h \sigma/x] \vdash \Psi}{\Gamma \mid \Phi, \nabla \sigma. \exists x:\tau. \varphi \vdash \Psi}$$

$$\begin{aligned} & \sigma \Vdash_{\rho, \Gamma} \nabla x: \iota. \exists y: \tau. \varphi \\ & \text{iff} \\ & \sigma \Vdash_{\rho, \Gamma} \exists h: \iota \rightarrow \tau. \nabla x: \iota. \varphi[h x/y] \end{aligned}$$

# Generic Judgements and Raising

Translate  $\sigma \triangleright \varphi$  to  $\nabla \sigma. \varphi$

$$\frac{\Gamma, h: \sigma \rightarrow \tau \mid \Phi, \nabla \sigma. \varphi[h \sigma / x] \vdash \Psi}{\Gamma \mid \Phi, \exists h: \sigma \rightarrow \tau. \nabla \sigma. \varphi[h \sigma / x] \vdash \Psi}$$

$$\begin{aligned} & \sigma \Vdash_{\rho, \Gamma} \nabla x: \iota. \exists y: \tau. \varphi \\ & \text{iff} \\ & \sigma \Vdash_{\rho, \Gamma} \exists h: \iota \rightarrow \tau. \nabla x: \iota. \varphi[h x / y] \end{aligned}$$

# Soundness & Completeness

## Soundness

Any derivable sequent is valid.

## Completeness

Any valid sequent  $\cdot \mid \Phi \vdash \Psi$  is derivable.

## Part II

Binding Structure,  $\nabla$  and  $\mathbb{N}$

# Binding Structure

Working with  $\alpha$ -equivalence classes is the same as working with freshly named instances.

# Binding Structure

Working with  $\alpha$ -equivalence classes is the same as working with freshly named instances.

A *category with binding structure* is a triple  $(\mathbb{B}, \otimes, V)$ ...

Category with finite limits

Monoidal Structure

Object  
(names)

$\mathbb{B}$  = FM sets (freshness)

$V$  =  $\mathbb{A}$

$A \otimes B$  =  $\{\langle a, b \rangle \in A \times B \mid a \# b\}$

# Binding Structure

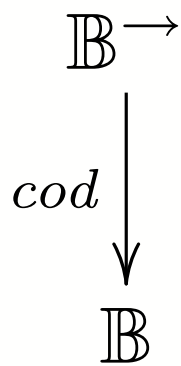
Working with  $\alpha$ -equivalence classes is the same as working with freshly named instances.

A *category with binding structure* is a triple  $(\mathbb{B}, \otimes, V)$  such that the functor  $W_V : f \mapsto f \otimes V$  is an equivalence of fibrations.

$$\begin{array}{ccc} \mathbb{B} \rightarrow & \xrightarrow{W_V} & \mathbb{B}/(- \otimes V) \\ & \searrow \text{cod} & \swarrow Gl(- \otimes V) \\ & \mathbb{B} & \end{array}$$

# Codomain Fibration

Working with  $\alpha$ -equivalence classes is the same as working with freshly named instances.

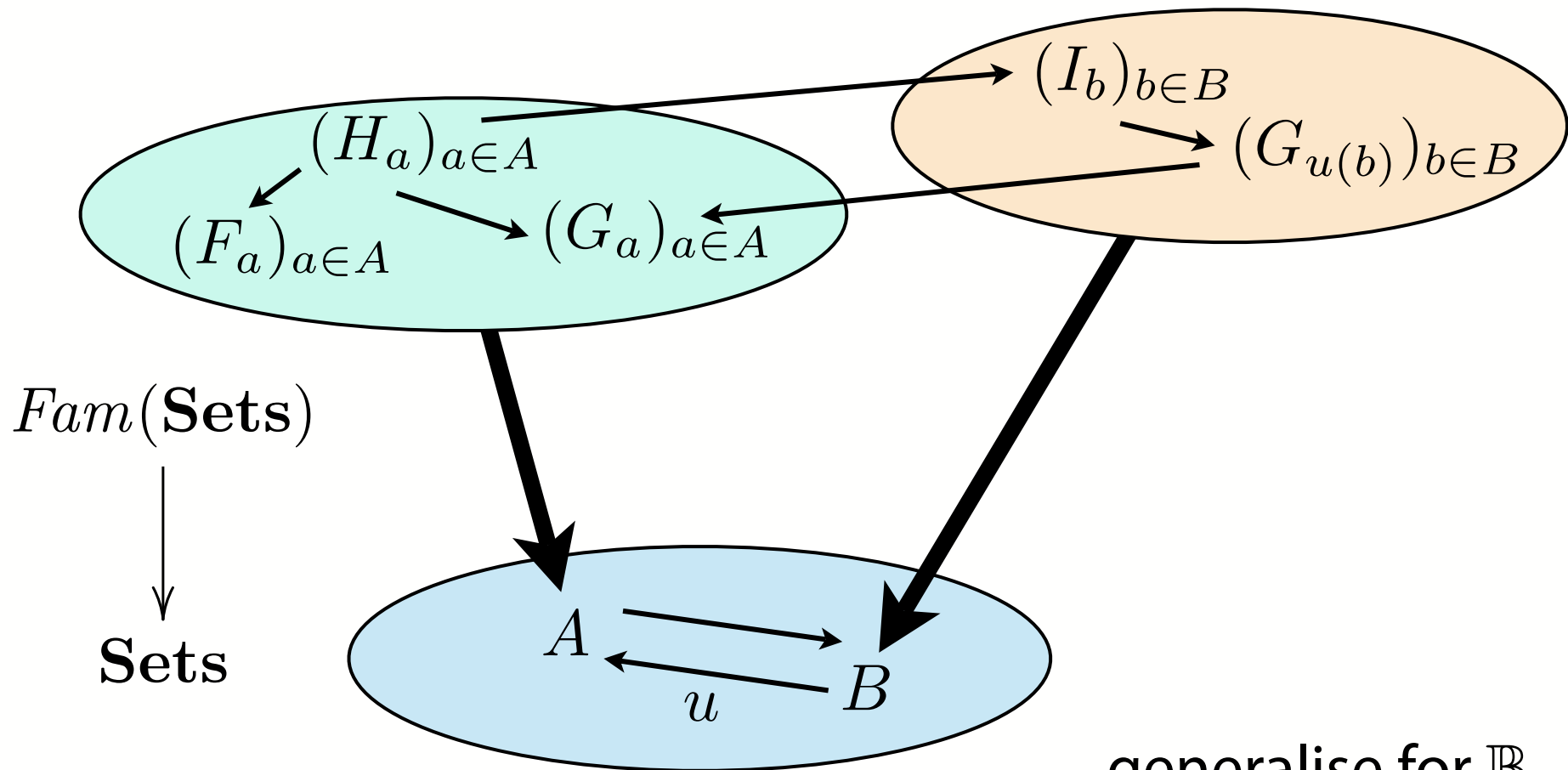


- Category with pullbacks  $\mathbb{B}$
- Families of  $\mathbb{B}$ -objects indexed by  $\mathbb{B}$ -objects
- Dependent type theory  $\Gamma \vdash A$

Other choices possible, e.g. subobject-logic

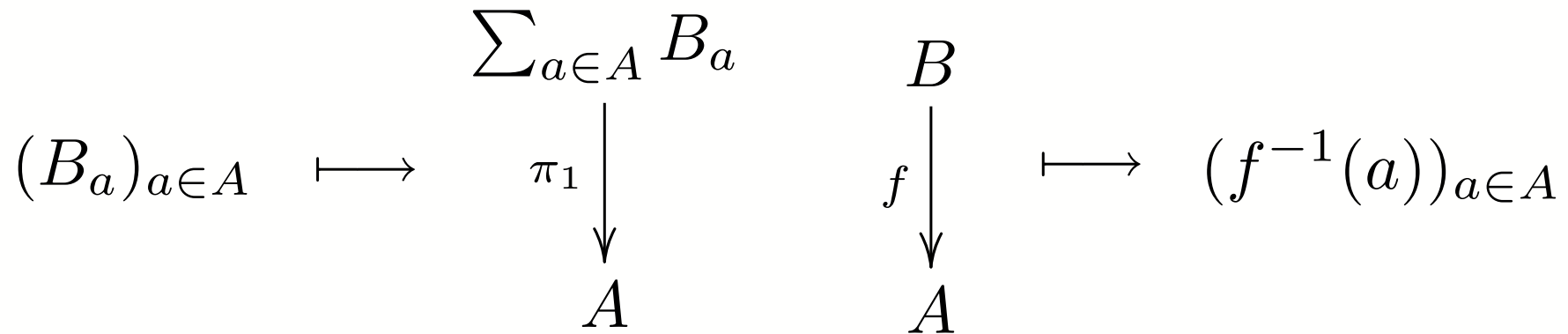
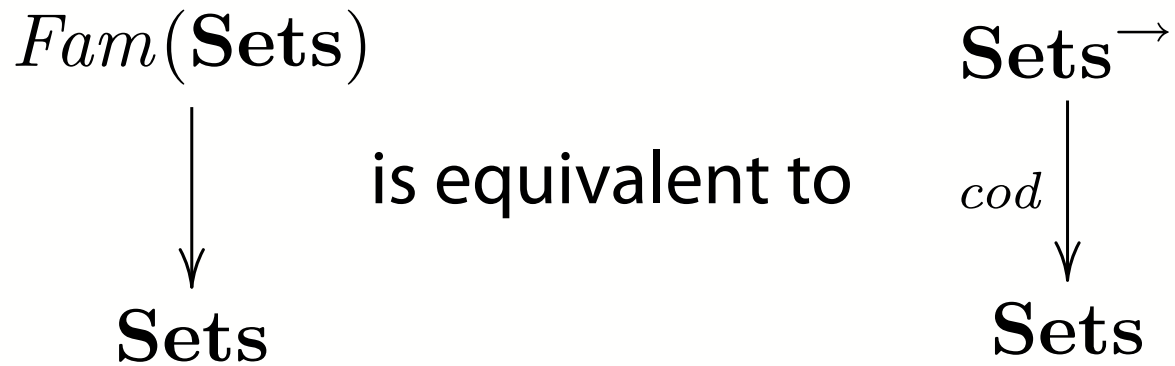
# Codomain Fibration

Families of sets

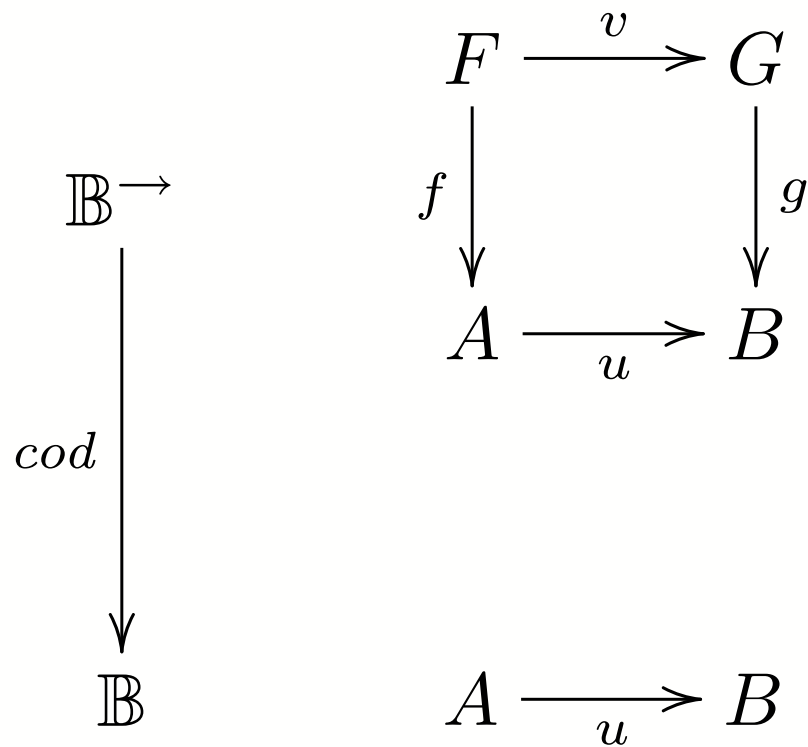


...generalise for  $\mathbb{B}$

# Codomain Fibration



# Codomain Fibration



**In Sets:**

$$(v_a : F_a \rightarrow G_{u(a)})_{a \in A}$$

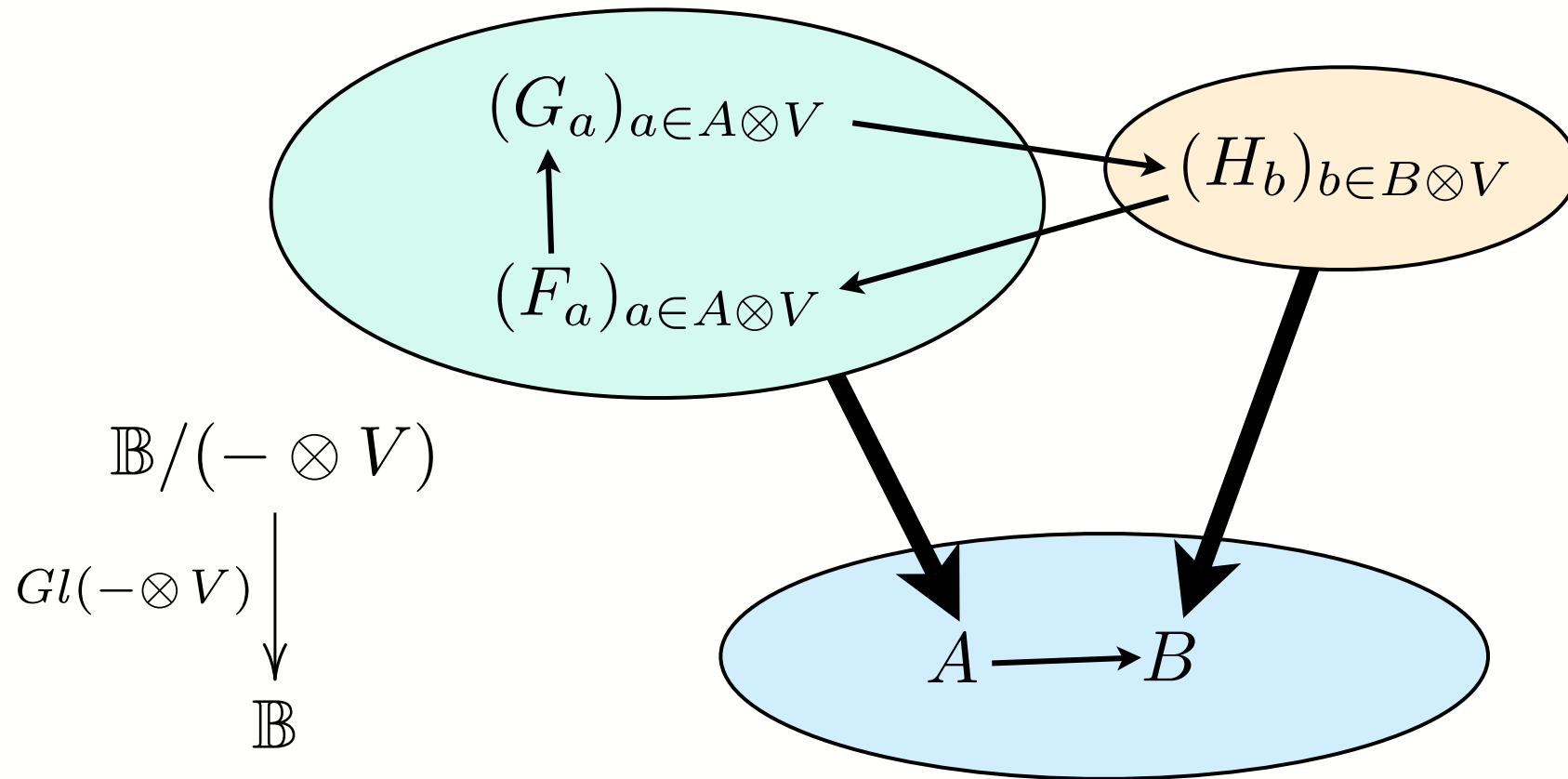
# Glueing a Fresh Name

Working with  $\alpha$ -equivalence classes is the same as working with freshly named instances.

$$\begin{array}{c} \mathbb{B}/(- \otimes V) \\ \text{Gl}(- \otimes V) \downarrow \\ \mathbb{B} \end{array}$$

- Families of  $\mathbb{B}$ -objects indexed by  $\mathbb{B}$ -objects of the form  $A \otimes V$
- Dependent type theory with judgements  $\Gamma \# v:V \vdash A$

# Glueing a Fresh Name



# Glueing a Fresh Name

$$\begin{array}{ccc} \mathbb{B}/(- \otimes V) & & \begin{array}{ccc} F & \xrightarrow{v} & G \\ f \downarrow & & \downarrow g \\ A \otimes V & \xrightarrow{u \otimes V} & B \otimes V \end{array} \\ \downarrow \text{Gl}(- \otimes V) & & \\ \mathbb{B} & & A \xrightarrow{u} B \end{array}$$

# Adding a Fresh Name

$$\begin{array}{ccc} \mathbb{B}^{\rightarrow} & \xrightarrow{W_V} & \mathbb{B}/(- \otimes V) \\ & \searrow \text{cod} & \swarrow Gl(- \otimes V) \\ & & \mathbb{B} \end{array}$$

# Adding a Fresh Name

$$\begin{array}{ccc} B & & B \otimes V \\ \downarrow f & \longmapsto & \downarrow f \otimes V \\ A & & A \otimes V \end{array}$$

$$\begin{array}{ccc} \mathbb{B} \rightarrow & \xrightarrow{W_V} & \mathbb{B}/(- \otimes V) \\ & \searrow \text{cod} & \swarrow Gl(- \otimes V) \\ & \mathbb{B} & \end{array}$$

# Adding a Fresh Name

$$(F_a)_{a \in A} \quad \longmapsto \quad (F_a \otimes \{v\})_{\langle a, v \rangle \in A \otimes V}$$

$$\begin{array}{ccc} \mathbb{B}^{\rightarrow} & \xrightarrow{W_V} & \mathbb{B}/(- \otimes V) \\ & \searrow \text{cod} & \swarrow Gl(- \otimes V) \\ & & \mathbb{B} \end{array}$$

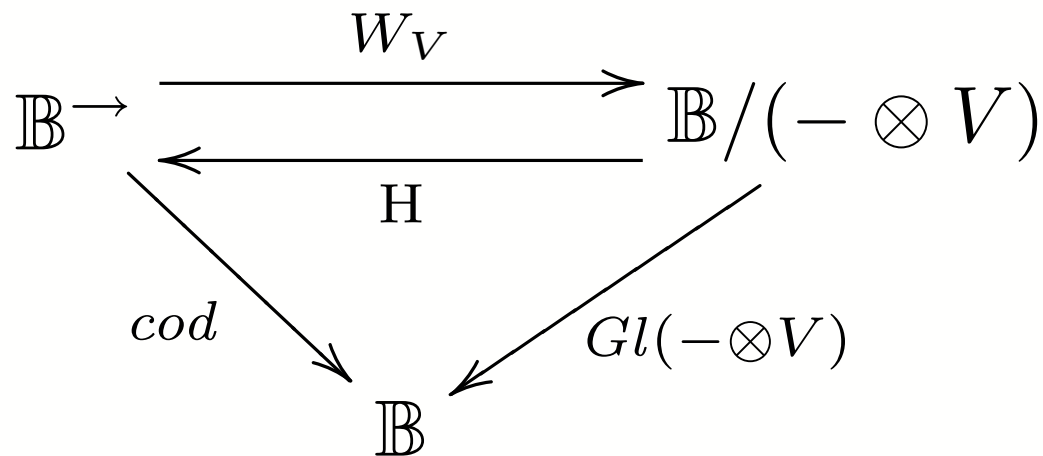
# Binding Structure

Working with  $\alpha$ -equivalence classes is the same as working with freshly named instances.

A *category with binding structure* is a triple  $(\mathbb{B}, \otimes, V)$  such that the functor  $W_V : f \mapsto f \otimes V$  is an equivalence of fibrations.

$$\begin{array}{ccc} \mathbb{B} \rightarrow & \xrightarrow{W_V} & \mathbb{B}/(- \otimes V) \\ & \searrow \text{cod} & \swarrow Gl(- \otimes V) \\ & \mathbb{B} & \end{array}$$

# The Equivalence



Natural vertical  
Isomorphisms

$$W_V \circ H \cong Id$$

$$H \circ W_V \cong Id$$

$W_V$  — Add a fresh name

$H$  — Bind a name (take  $\alpha$ -equivalence class)

$$\text{FM sets: } W_V (F_a)_{a \in A} = (F_a \otimes \{v\})_{\langle a, v \rangle \in A \otimes V}$$

$$H (G_{\langle a, v \rangle})_{\langle a, v \rangle \in A \otimes V} = ([v \in V] G_{\langle a, v \rangle})_{a \in A}$$

# Binding Structure implies...

... FM-like constructs  $\mathcal{N}$ ,  $[\Delta]X$ ,  $n.x$ ,  $x@n$ ,  $new\ n.x$

[Gabbay & Pitts]

$H \dashv W_V \dashv H$  with  $\eta' = \varepsilon^{-1}$  and  $\varepsilon' = \eta^{-1}$

# Binding Structure implies...

... FM-like constructs  $\mathcal{V}$ ,  $[\Delta]X$ ,  $n.x$ ,  $x@n$ ,  $new\ n.x$   
[Gabbay & Pitts]

$\mathbb{H} \dashv W_V \dashv \mathbb{H}$  with  $\eta' = \varepsilon^{-1}$  and  $\varepsilon' = \eta^{-1}$

$$\mathbb{H}0 \cong 0$$

$$\mathbb{H}1 \cong 1$$

$$\mathbb{H}(A + B) \cong \mathbb{H}A + \mathbb{H}B$$

$$\mathbb{H}(A \times B) \cong \mathbb{H}A \times \mathbb{H}B$$

$$\mathbb{H}(A \Rightarrow B) \cong \mathbb{H}A \Rightarrow \mathbb{H}B$$

# H — New-Name Quantifier

H restricts to a some/any quantifier

$$\frac{\Gamma \# v: V \mid \varphi \# v \vdash \psi}{\Gamma \mid \varphi \vdash \text{H}v:V. \psi}$$

$$\frac{\Gamma \# v: V \mid \varphi \vdash \psi \# v}{\Gamma \mid \text{H}v:V. \varphi \vdash \psi}$$

Under certain circumstances:

$$\frac{\Gamma \# v: V \mid \varphi \vdash \psi}{\Gamma \mid \varphi \vdash \text{H}v:V. \psi}$$

$$\frac{\Gamma \# v: V \mid \varphi \vdash \psi}{\Gamma \mid \text{H}v:V. \varphi \vdash \psi}$$

— the rules for  $\mathcal{H}$ !

# H — Abstraction Set

## Binding

Unit  $\eta$  of  $H \dashv W_V$

$$\langle v, x \rangle \longmapsto v.x \# v$$

## Concretion

Counit  $\varepsilon$  of  $W_V \dashv H$

$$y \# v \longmapsto y@v$$

## Equations

$\eta' = \varepsilon^{-1}$  and  $\varepsilon' = \eta^{-1}$

$$(v.x)@v = x$$

$$v.(y@v) = y$$

# Binding Structure and $\text{FO}\lambda^{\nabla}$

## Interpretation of Types

As in Part I

$A\sigma$  — Family of sets

$(-)[\theta]: A\sigma \rightarrow A\sigma'$  — Substitution action

## Interpretation of the Logic

Internal logic of a category with binding structure

$\nabla$  — New-quantifier  $\mathbb{H}$

Raising — Binding for  $\mathbb{H}$

# Binding Structure and $\nabla$

Binding Structure	FO $\lambda^{\nabla}$ -Interpretation
Linear Species	Part I
Species	Part I $+ \nabla x. \nabla y. \varphi \Leftrightarrow \nabla y. \nabla x. \varphi$
FM Sets	[Gabbay & Cheney '04] $\nabla x. \varphi = \forall x. \varphi[n(x)/x]$
?	[Miculan & Yemane '05]

# Conclusion

$\nabla$  and  $\mathcal{N}$  are both  $\mathbb{H}$ , in different binding structures

## Further Work

- Simple Semantics for Intuitionistic  $\text{FO}\lambda^\nabla$
- Proof theory of  $\text{FO}\lambda^\nabla$  for Nominal Logic
- Relation to [Miculan & Yemane '05]
- a-Logic [Gabbay & Gabbay '05]