Interaction Semantics and Programming Language Compilation

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Introduction

Interaction Semantics builds mathematical models for programming languages from interacting processes.

Such models can help understand low-level decompositions of high-level languages.

```
let rec fib x =
     if x < 1 then 1 else (fib (x - 1)) + (fib (x - 2))
```

```
...
%x6 = phi i32 [ 38, %case1 ], [ %unpack35, %case145 ],
                [ %add, %case167 ]
%add = add i32 %x6, -1
%eq47 = icmp ne i32 %add, 0
switch i1 %eq47, label %case049 [ i1 true, label %case148 ]
...
```

Introduction

Need better understanding for:

- formal verification
- compositional reasoning
- resource usage analysis and certification
- modularity

Game Semantics for Logic

Explain logic in terms of dialogues between disputing parties.

Proponent and Opponent argue about a proposition:

- Proponent tries to defend it.
- Opponent tries to refute it.
- The logic defines the mode of interaction.
  - How can a formula be attacked?
  - How can a formula be defended?

A proof is a strategy for Proponent to defend the proposition against any possible attack.

Game Semantics for Constructive Logic

[Lorenzen & Lorenz, 1950s]

$$(\bot \wedge \varphi) \vee \top$$

Opponent **Proponent**

Which of the disjuncts is true?

Game Semantics for Constructive Logic

[Lorenzen & Lorenz, 1950s]

$$(\bot \wedge \varphi) \vee \top$$

Opponent

Proponent

Which of the disjuncts is true?

The left one $\bot \wedge \varphi$ *is true.*

Game Semantics for Constructive Logic

[Lorenzen & Lorenz, 1950s]

$$(\bot \wedge \varphi) \vee \top$$

Opponent

Proponent

Which of the disjuncts is true?

The left one $\bot \wedge \varphi$ *is true.*

Then explain why $\bot$ *is true.*

Game Semantics for Programs

Proponent now defends the claim:

I have a program of type X.

Attacks become requests for information.

Programs are modelled by strategies that explain how Proponent can answer any request for information.

Game Semantics for Programs

$\texttt{int} \to \texttt{int}$

Opponent | **Proponent**

What does your function return?

Game Semantics for Programs

$$\texttt{int} \rightarrow \texttt{int}$$

Opponent | **Proponent**

What does your function return?

What is the function argument?

Game Semantics for Programs

$\texttt{int} \rightarrow \texttt{int}$

Opponent | **Proponent**

What does your function return?

What is the function argument?

The argument is 5.

Game Semantics for Programs

$$\texttt{int} \to \texttt{int}$$

Opponent

Proponent

What does your function return?

What is the function argument?

The argument is 5.

Then the function returns 6.

Game Semantics for Programs

The strategy of a program derives from strategies of its parts.

$$\lambda x.\, x + 1 \colon \texttt{int} \to \texttt{int}$$

$$x + 1 \colon \texttt{int}$$

$$x \colon \texttt{int} \qquad\qquad 1 \colon \texttt{int}$$

Game Semantics for Programs

The strategy of a program derives from strategies of its parts.

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Game Semantics for Programs

The strategy of a program derives from strategies of its parts.

What does the function return?

$$\lambda x.\, x + 1 \colon \texttt{int} \to \texttt{int}$$

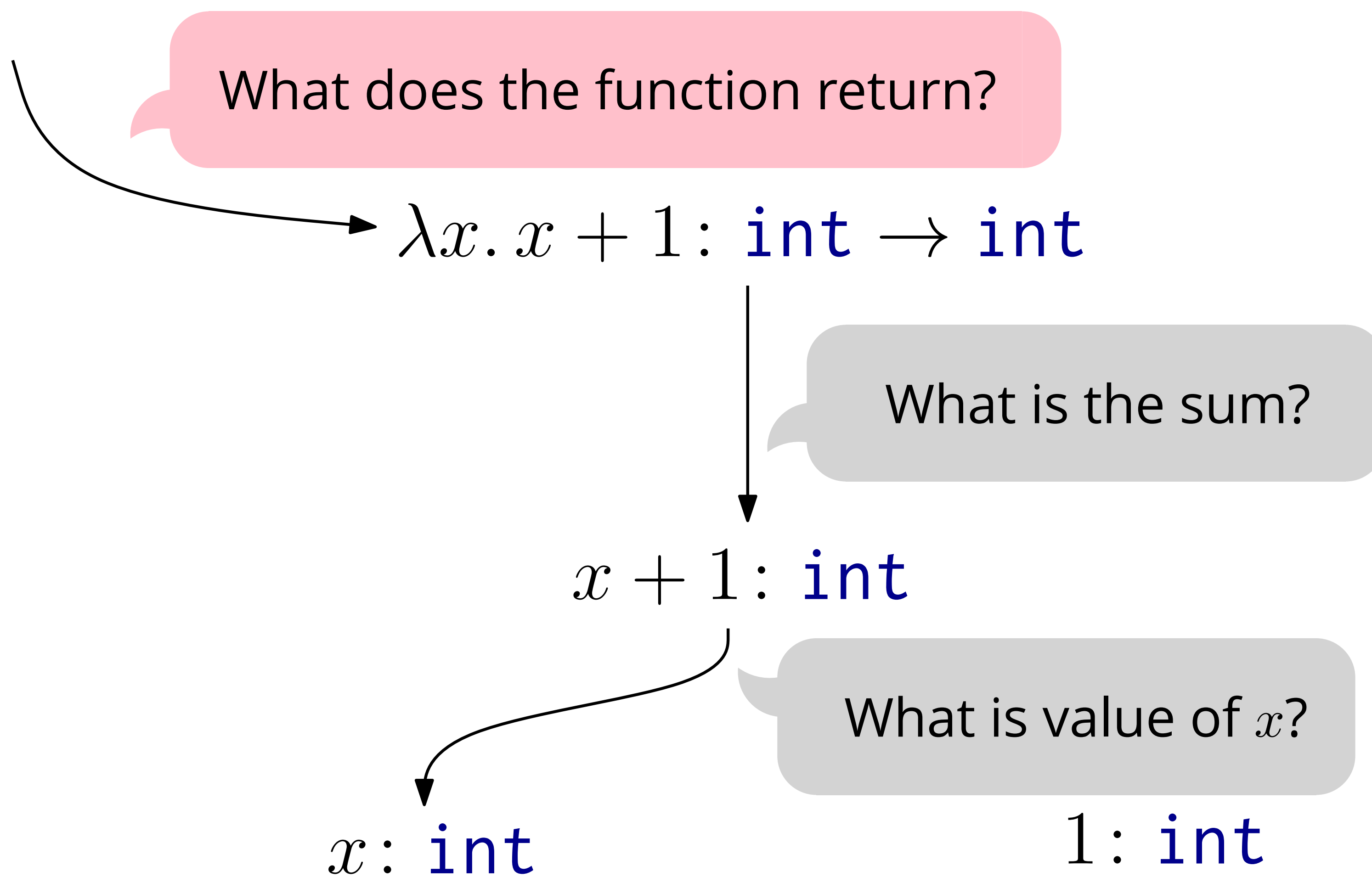
What is the sum?

$$x + 1 \colon \texttt{int}$$

$x : \texttt{int}$ $\qquad$ $1 \colon \texttt{int}$

Game Semantics for Programs

The strategy of a program derives from strategies of its parts.



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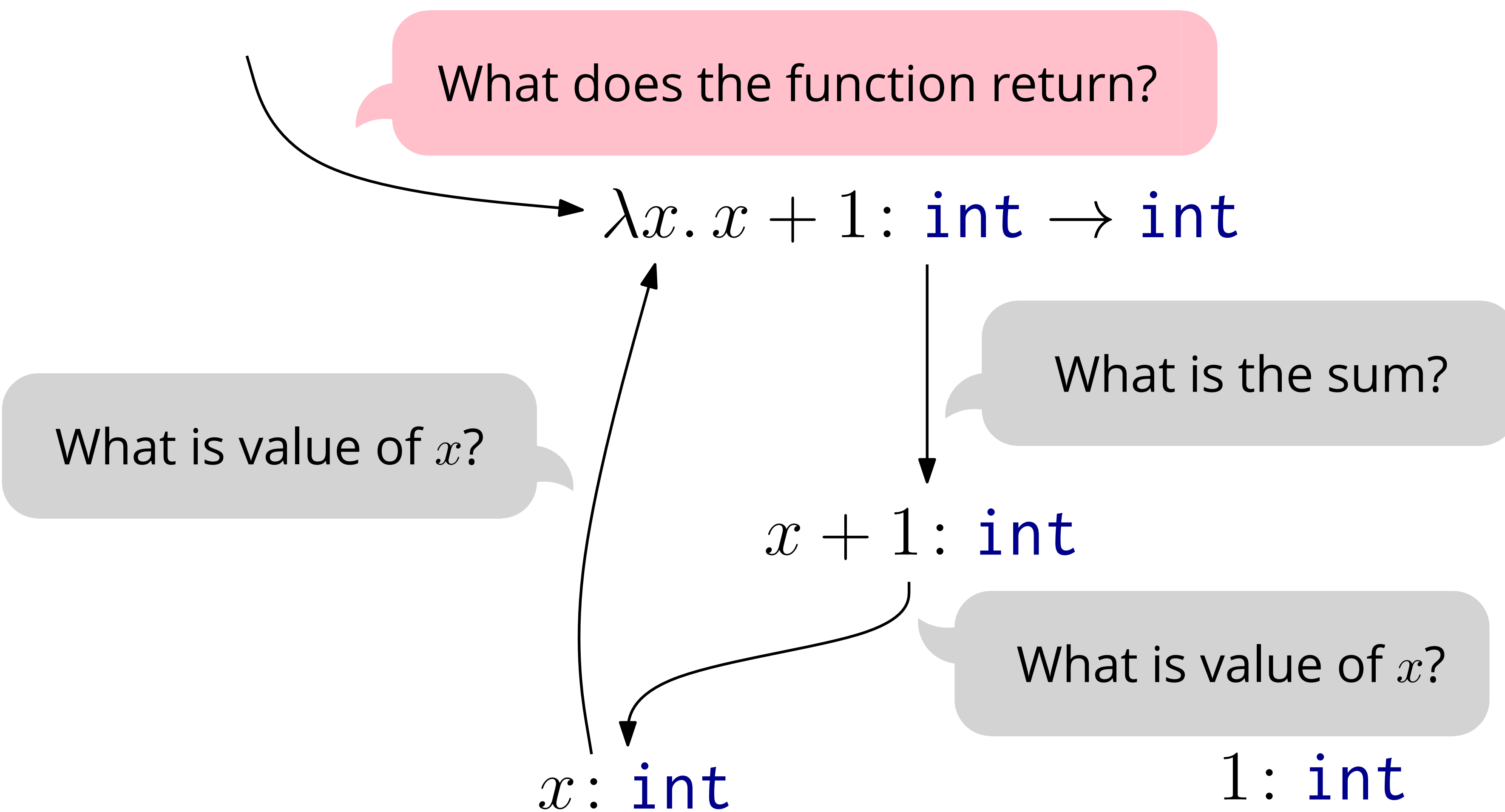

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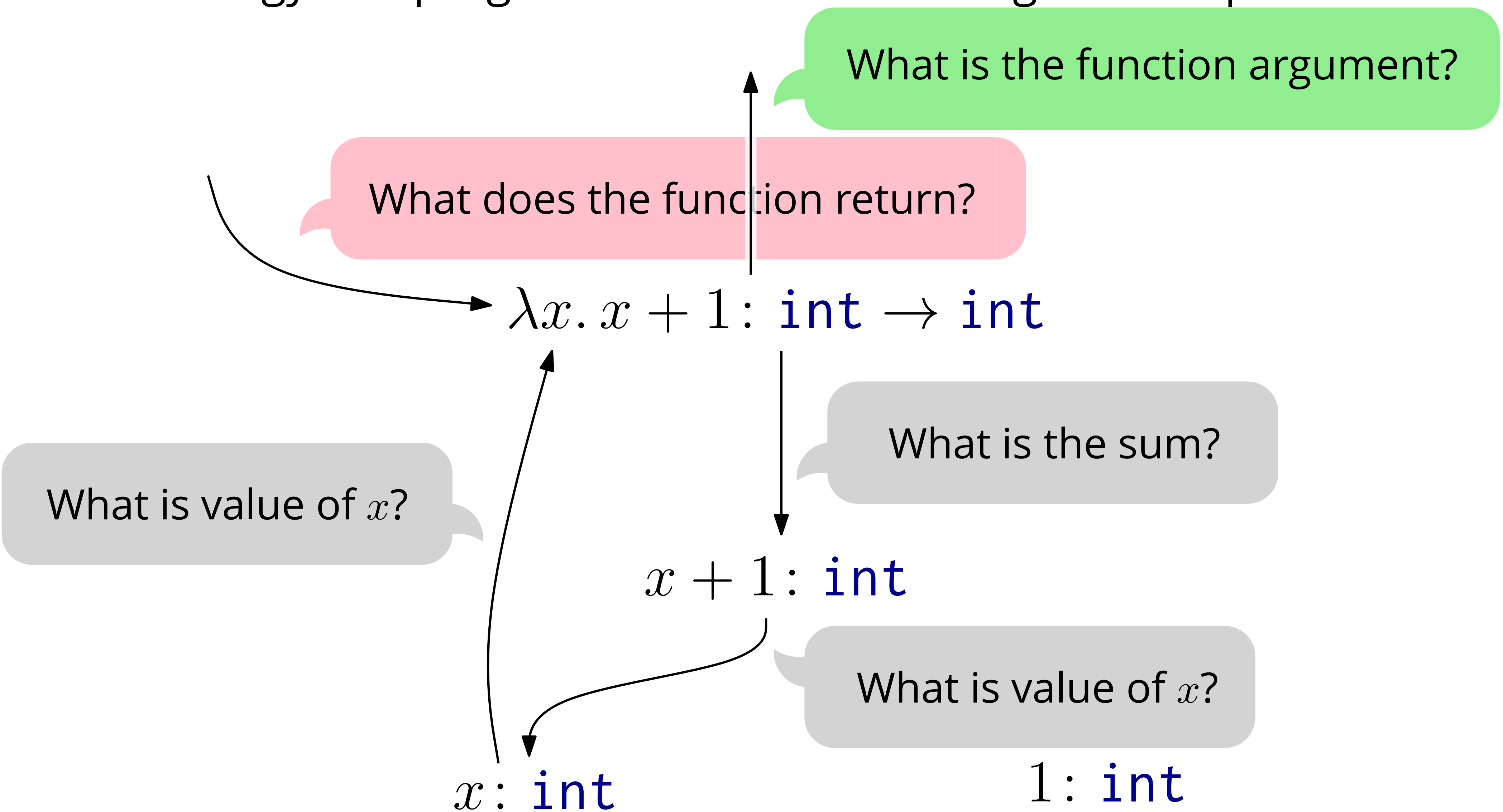

Game Semantics for Programs

The strategy of a program derives from strategies of its parts.

The argument is 5.

$$\lambda x.\, x + 1 \colon \texttt{int} \to \texttt{int}$$

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$$x \colon \texttt{int} \qquad 1 \colon \texttt{int}$$

Game Semantics for Programs

The strategy of a program derives from strategies of its parts.

The argument is 5.

$$\lambda x.\, x + 1 \colon \texttt{int} \to \texttt{int}$$

The value is 5.

$$x + 1 \colon \texttt{int}$$

$$x \colon \texttt{int} \qquad\qquad 1 \colon \texttt{int}$$

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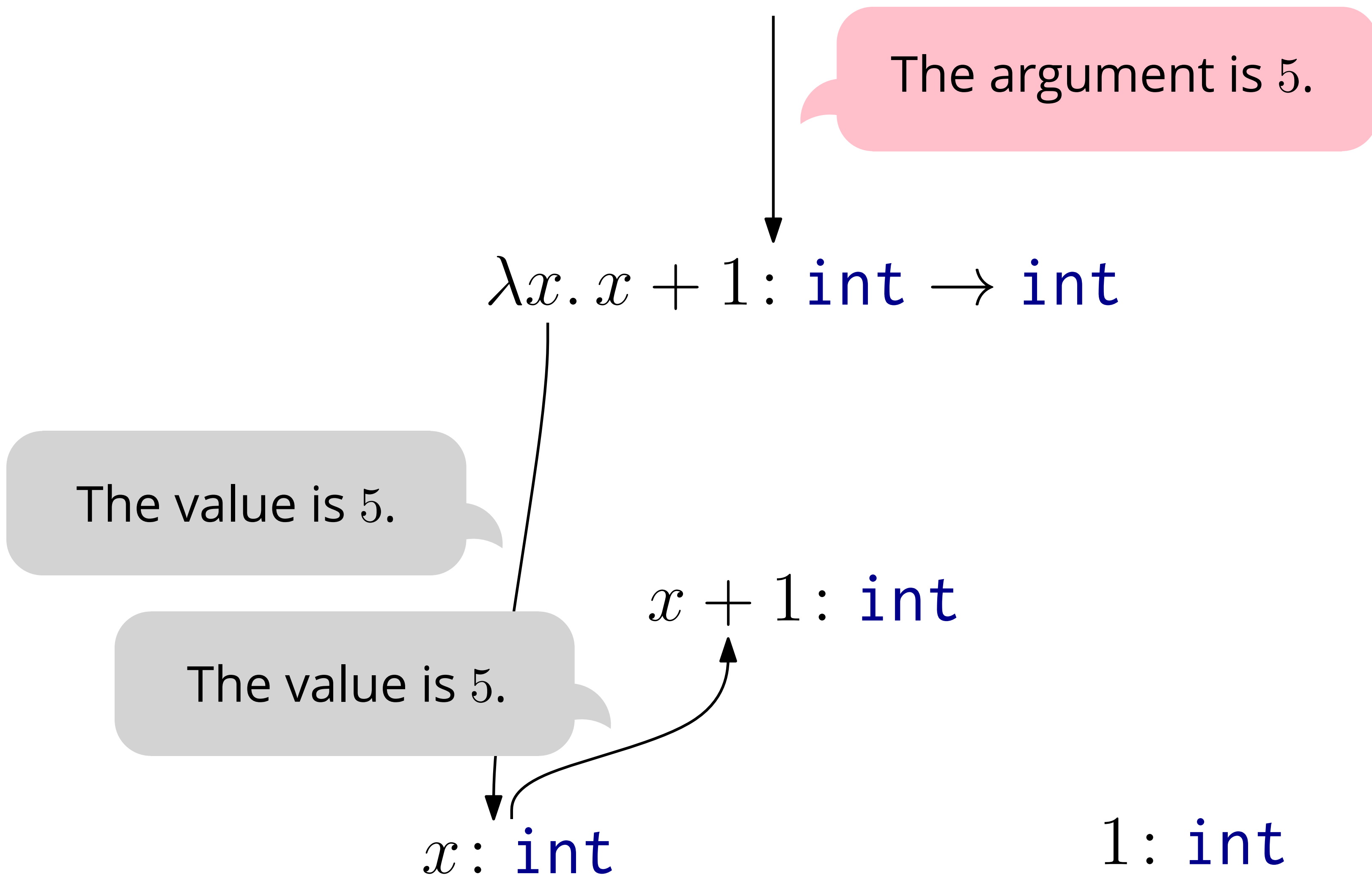

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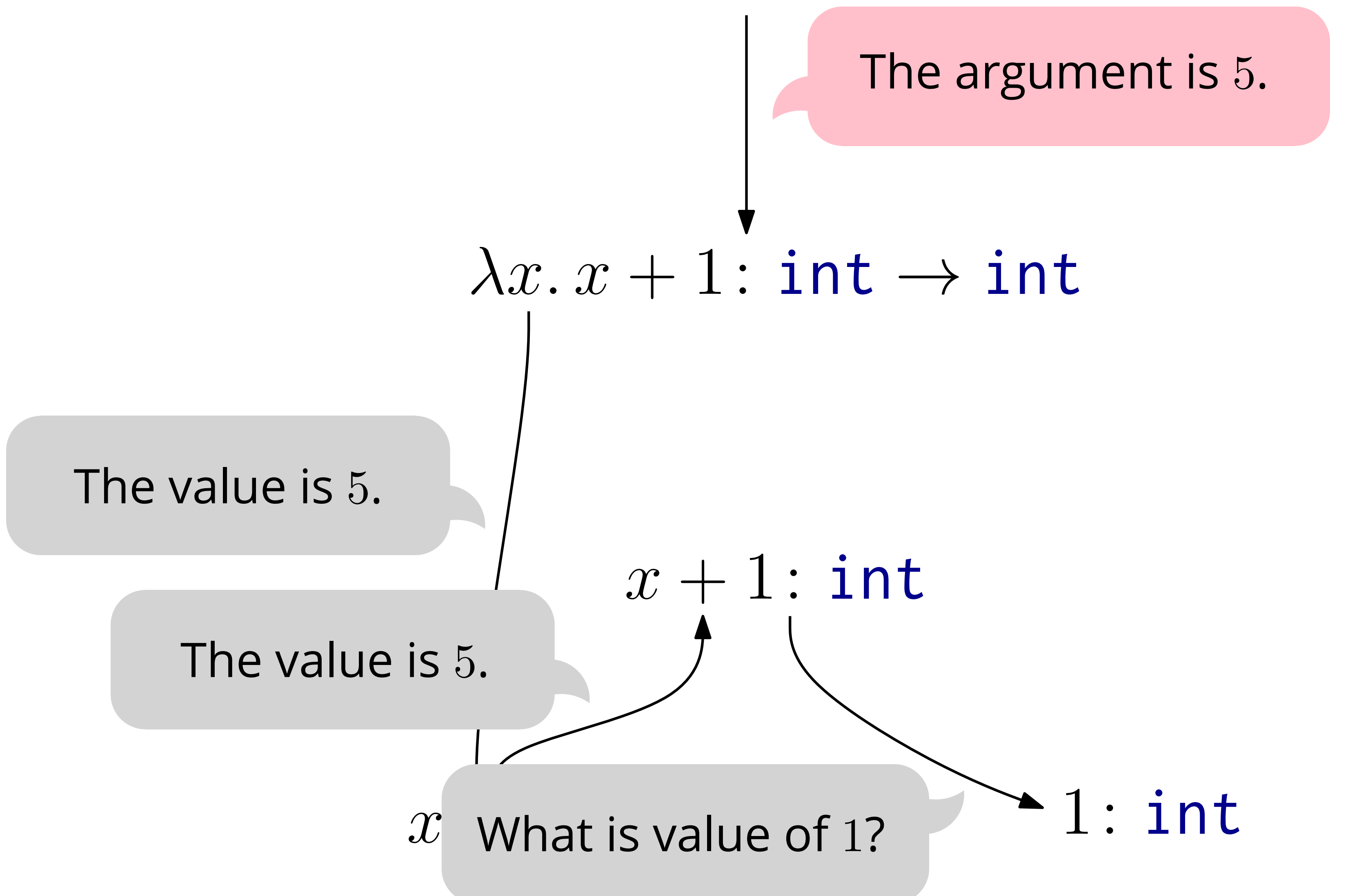

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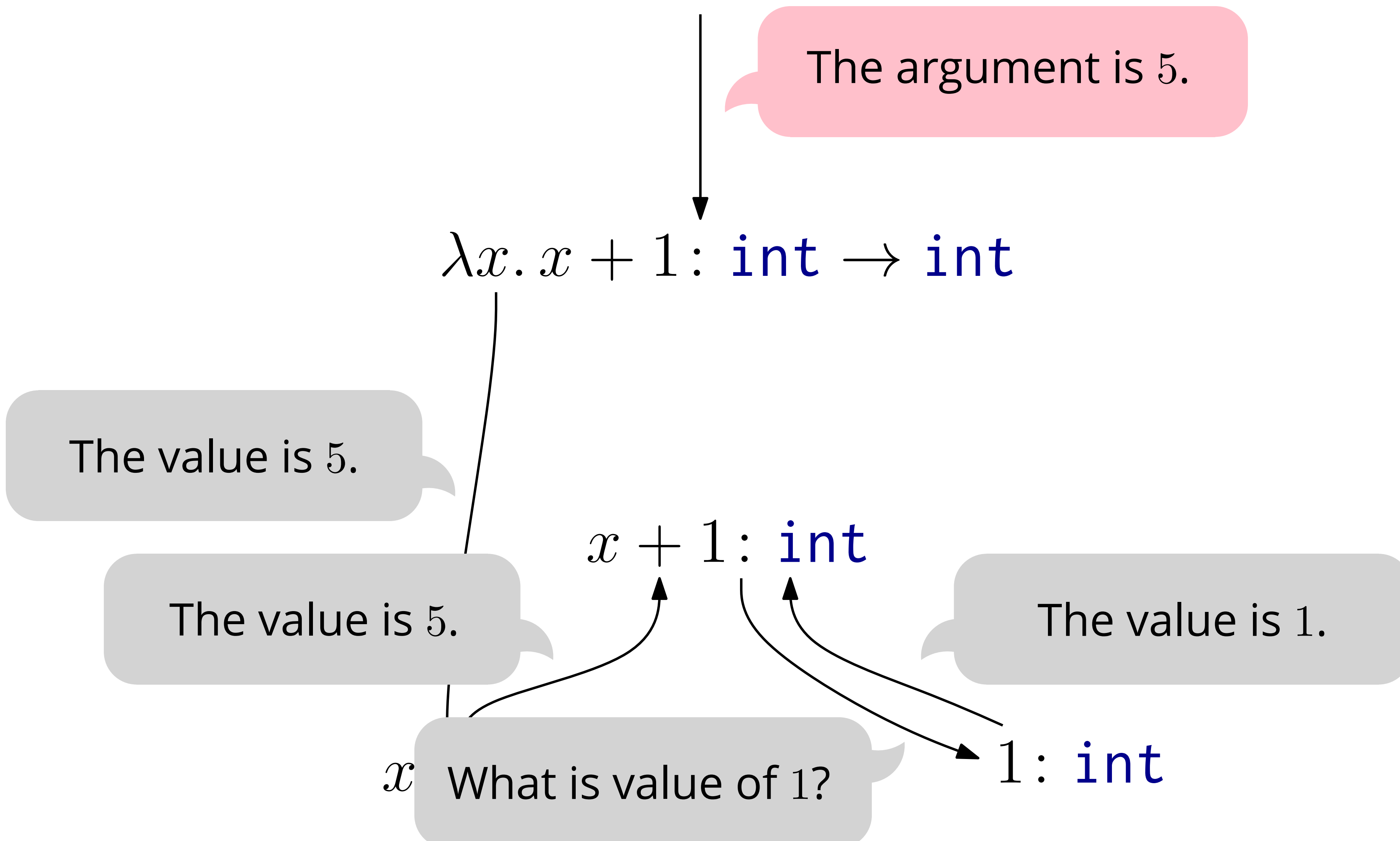

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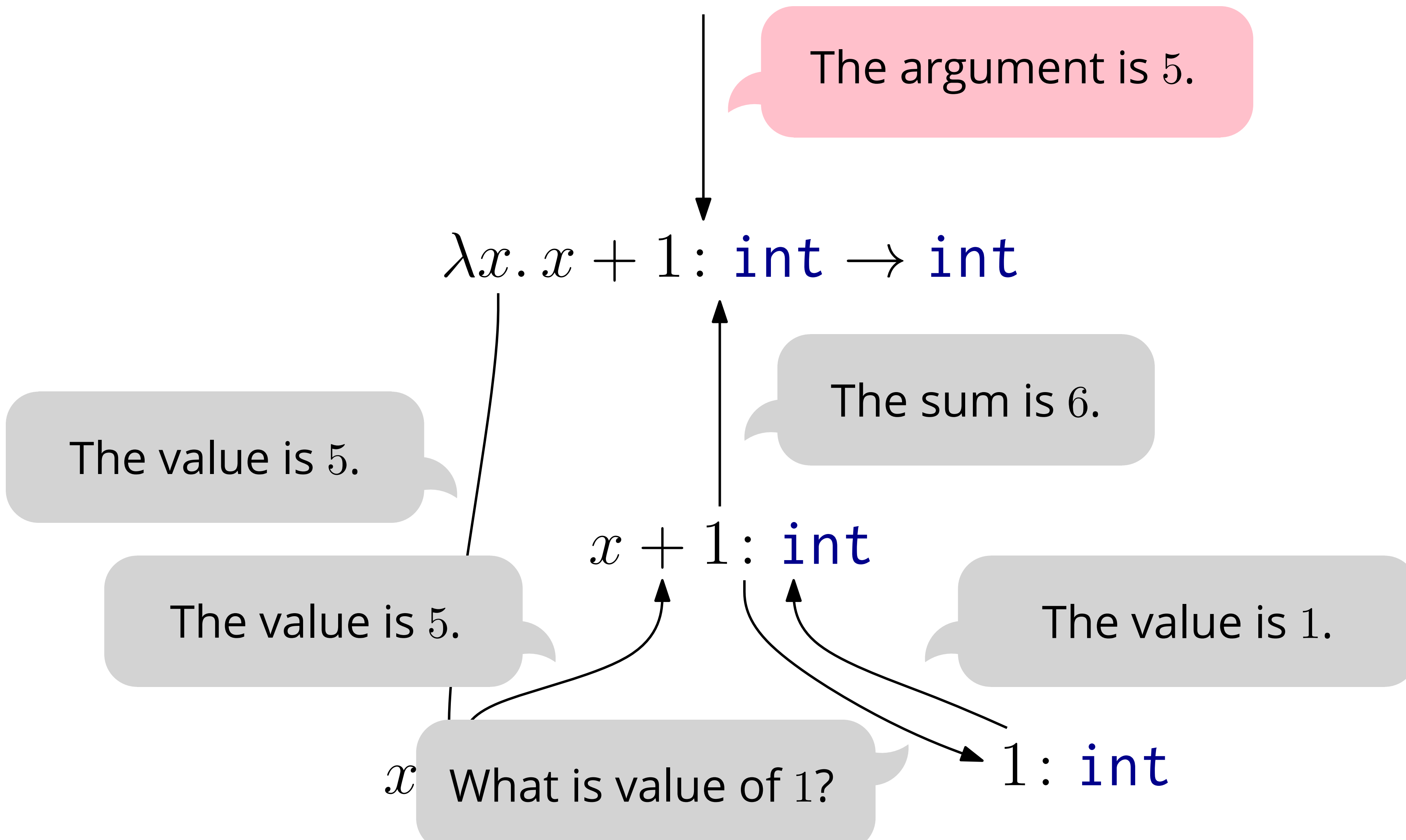

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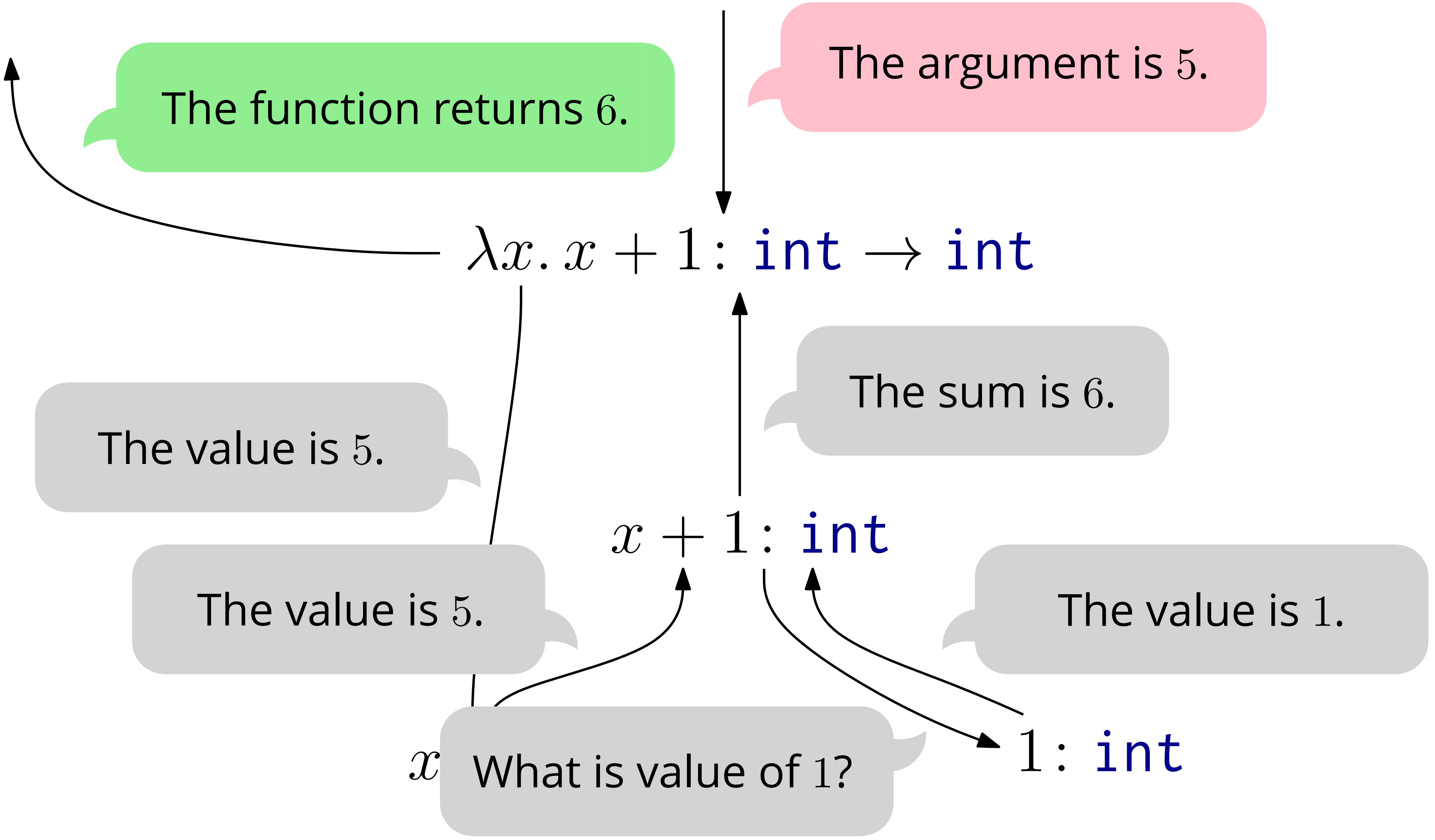


Structure in Game Semantics

Game semantics has developed a number of mathematical constructions that turn a very simple model of interaction dialogues into precise models of many programming languages.

Fully abstract model for PCF

- [Hyland & Ong, 1994]
- [Abramsky, Jagadeesan & Malacaria, 1994]
- [Nickau 1994]

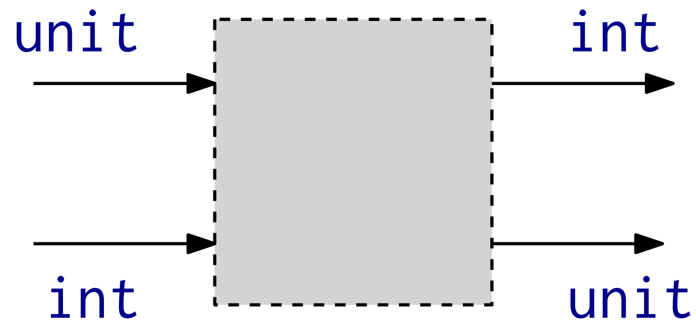
Geometry of Interaction

- closely related, with proof-theoretic motivation [Girard 1987]

Computation by Interaction

Implement programs by implementing their interaction strategies.

$$\texttt{int} \to \texttt{int}$$



Computation by Interaction

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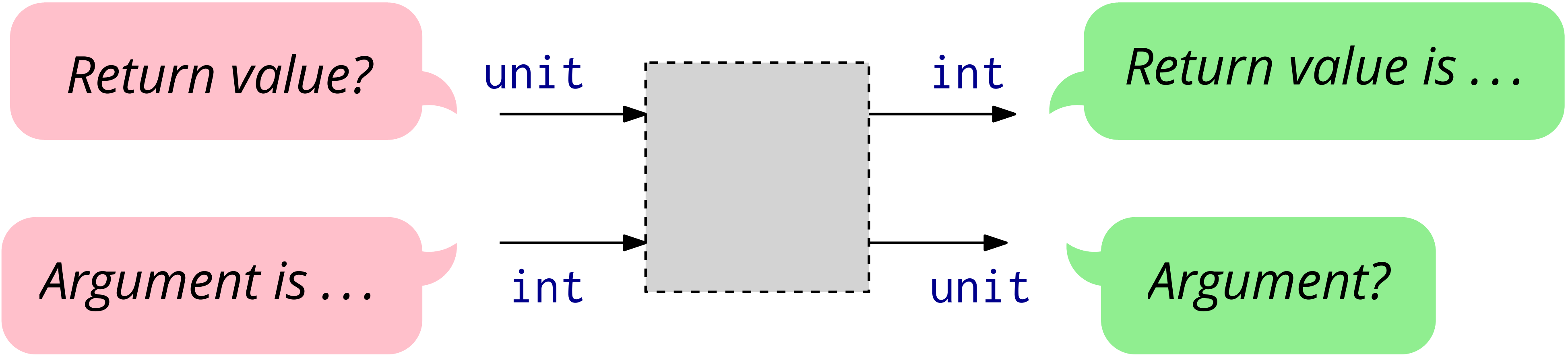
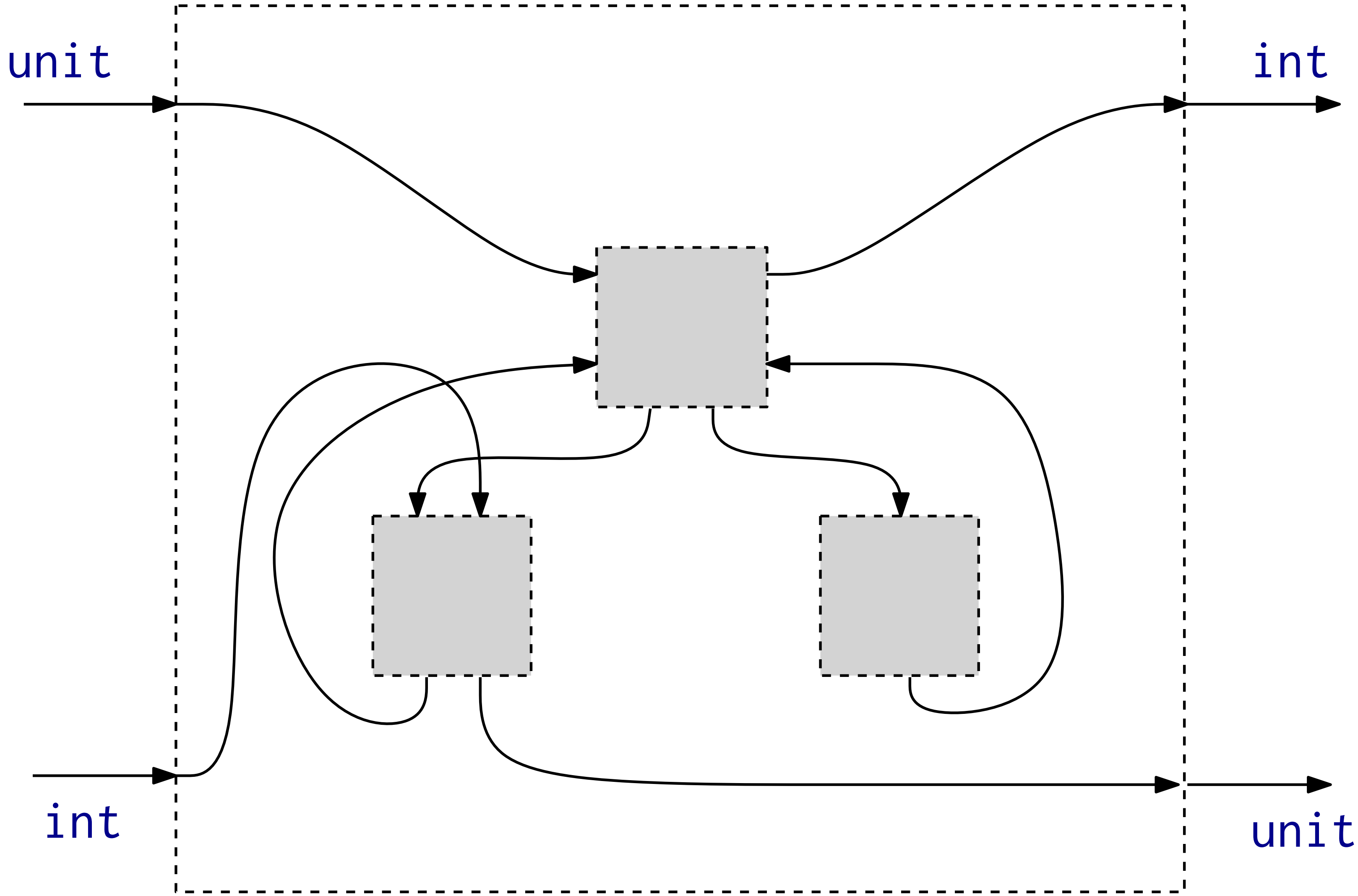


Computation by Interaction

Strategies are compositional building blocks.



Computation by Interaction

Construct a game semantic model from low-level programs.
Interpretation becomes compilation.

Developed for compilation to . . .

- abstract machines [Mackie, 1995]
- hardware circuits [Ghica, Smith & Singh, 2007]
- LOGSPACE Turing Machines [S., 2006], [Dal Lago & S., 2010]
- π-calculus [Honda, Yoshida & Berger, 2001]
- distributed processes [Fredriksson & Ghica, 2013]
- quantum circuits
  [Hoshino, Hasuo, Yoshimizu, Faggian, Dal Lago, 2014]
- . . .

Introduction

Consider interaction as a general approach to connect mathematical semantics to compiler construction.

Semantics

- mathematical structure
- compositionality
- proofs

Compiler Construction

- efficiency
- optimisations
- implementation

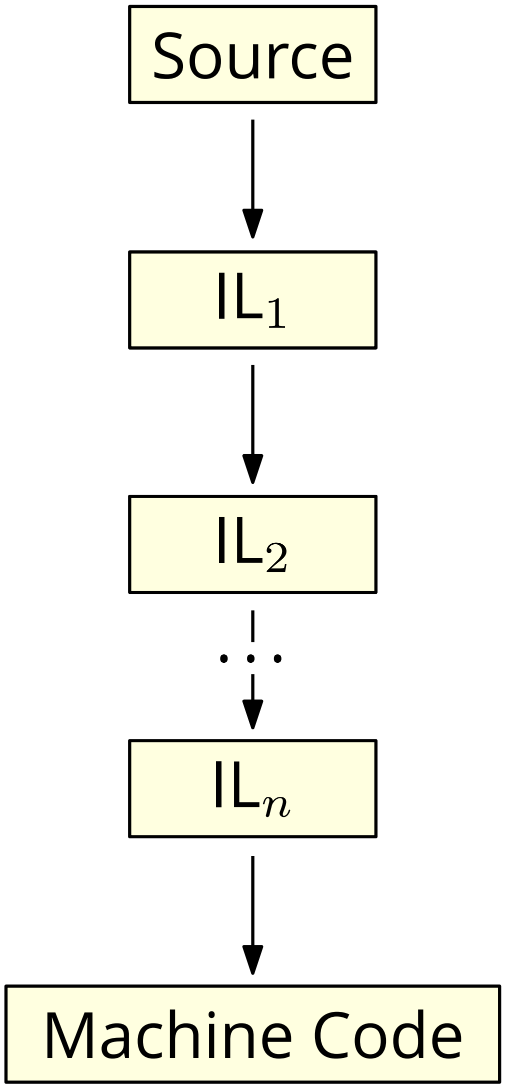
Overview

We look at the compilation of higher-order functional programming languages.

Compilers work by translating the source into a number of intermediate languages.

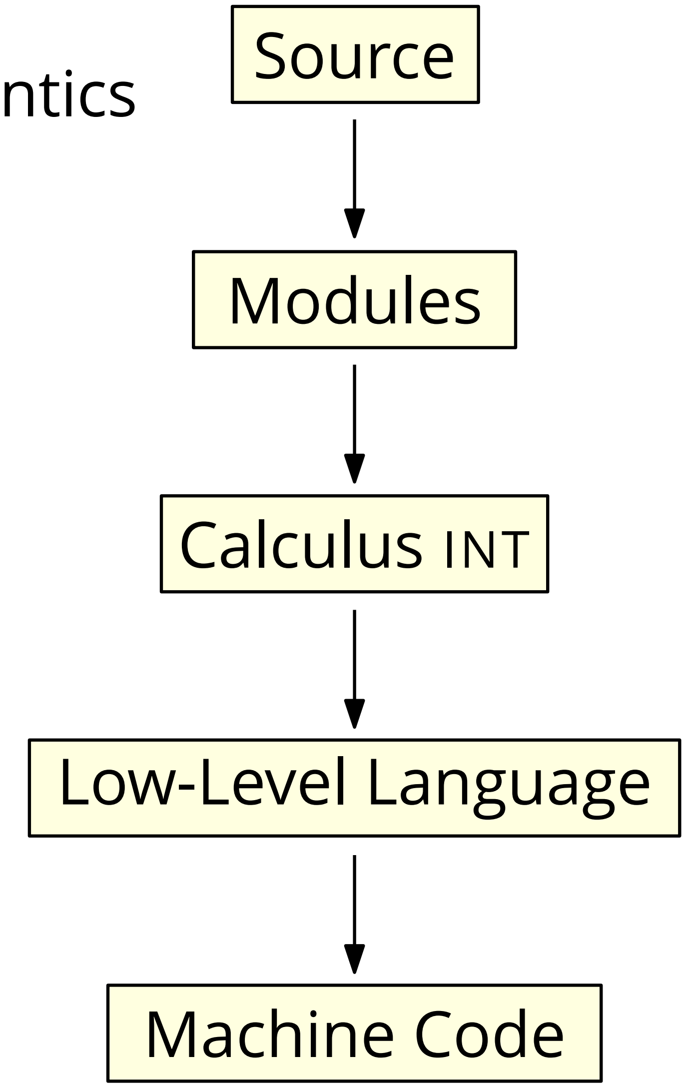
- decreasing level of abstraction
- optimisations at different levels

We use constructions from interaction semantics to construct a series of intermediate languages.



Overview

- Low-Level Programs
- Organising Low-Level Programs
  - Constructions from Interaction Semantics
  - Calculus INT
  - Simple Module System
- Compilation
  - Call-by-Name
  - Call-by-Value
- Relation to Defunctionalisation



Low-Level Programs

Low-Level Programs

Most compilers abstract from machine details by translating to an architecture-independent low-level language that is then translated to machine code.

Example: LLVM compiler infrastructure

- used by many compilers (Clang, Rust, . . .)
- portable assembler in static single assignment form (simple instructions, jumps, machine calls)
- compiler for many architectures

Low-Level Programs

LLVM IR

```
entry:
  ; initial value = 1.0 (inlined into phi)
  br label %loop

loop:       ; preds = %loop, %entry
  %i = phi double [ 1.000000e+00, %entry ], [ %nextvar, %loop ]
  ; body
  %calltmp = call double @putchard(double 4.200000e+01)
  ; increment
  %nextvar = fadd double %i, 1.000000e+00

  ; termination test
  %cmptmp = fcmp ult double %i, %n
  %booltmp = uitofp i1 %cmptmp to double
  %loopcond = fcmp one double %booltmp, 0.000000e+00
  br i1 %loopcond, label %loop, label %afterloop

afterloop:      ; preds = %loop
  ; loop always returns 0.0
  ret double 0.000000e+00
```

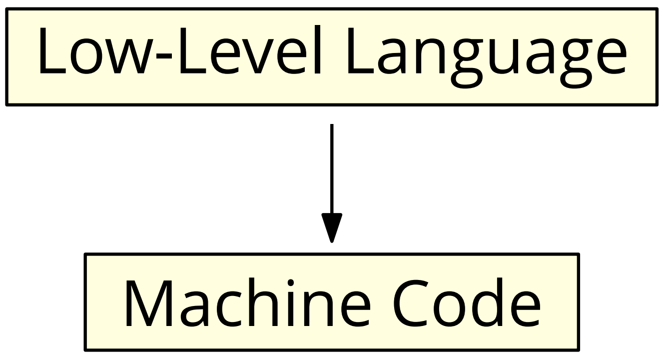
(Source: `http://llvm.org/docs/tutorial/LangImpl05.html`)

Low-Level Programs

We define a simple low-level language:

- similar abstraction level as LLVM assembly
- idealised heap (recursive types)
- functional presentation of static single assignment form

Similar languages are used in production compilers, e.g. Swift Intermediate Language.



Values and Types

Types

$$A, B \ ::= \ \alpha \mid \mathtt{int} \mid \mathtt{unit} \mid A \times B \mid 0 \mid A + B \mid \mu\alpha.A$$

Values

$$v, w ::= () \mid n \mid (v, w) \mid \mathtt{inl}(v) \mid \mathtt{inr}(v) \mid \mathtt{fold}(v)$$

Use algebraic data types as syntactic sugar for $\mu\alpha.A$.

Example: Write

$$\begin{aligned} \mathtt{type\ list}\langle\alpha\rangle \ &= \mathtt{Nil}\ \text{of}\ \mathtt{unit} \\ &\mid \text{Cons of}\ \alpha \times \mathtt{list}\langle\alpha\rangle \end{aligned}$$

for $\mu\beta.\,\mathtt{unit} + \alpha \times \beta$, where $\mathtt{Nil} = \mathtt{fold}(\mathtt{inl}())$ and $\text{Cons}(h, t) = \mathtt{fold}(\mathtt{inr}(h, t))$.

Blocks

Programs are constructed from blocks.

A **block** has the form

$$\texttt{label}(x : A) = \mathit{body}$$

where

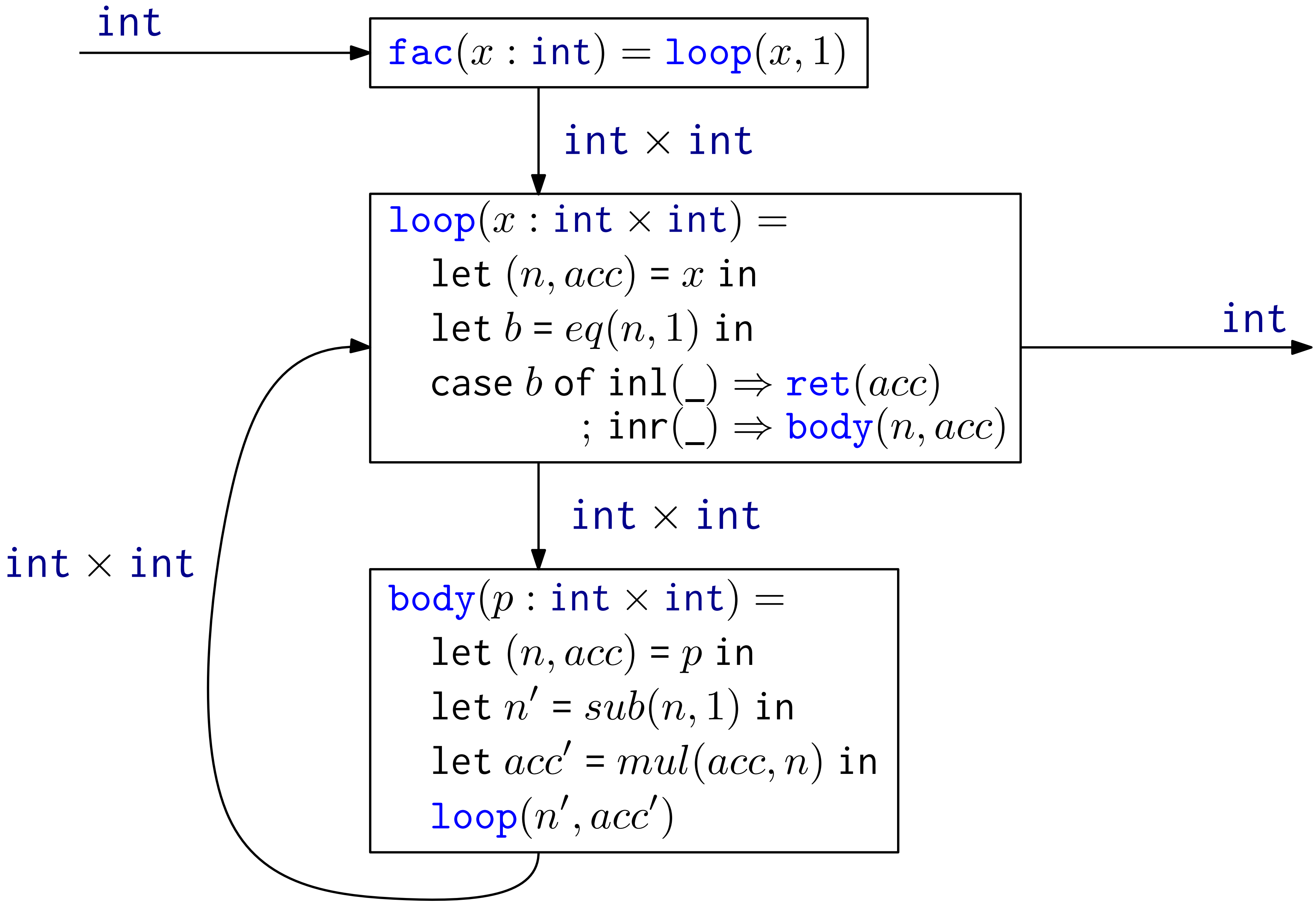
$$\begin{aligned}
\mathit{body} \;::=\; & \texttt{let } x = \texttt{primop}(v) \texttt{ in } \mathit{body} \\
\mid\; & \texttt{let } (x, y) = v \texttt{ in } \mathit{body} \\
\mid\; & \texttt{let fold}(x) = v \texttt{ in } \mathit{body} \\
\mid\; & \texttt{label}(v) \\
\mid\; & \texttt{case } v \texttt{ of inl}(x) \to \texttt{label}_1(v_1)\,;\ \texttt{inr}(y) \to \texttt{label}_2(v_2)
\end{aligned}$$

`primop` ranges over primitive operations, such as `add`, `mul`, or `syscall`,

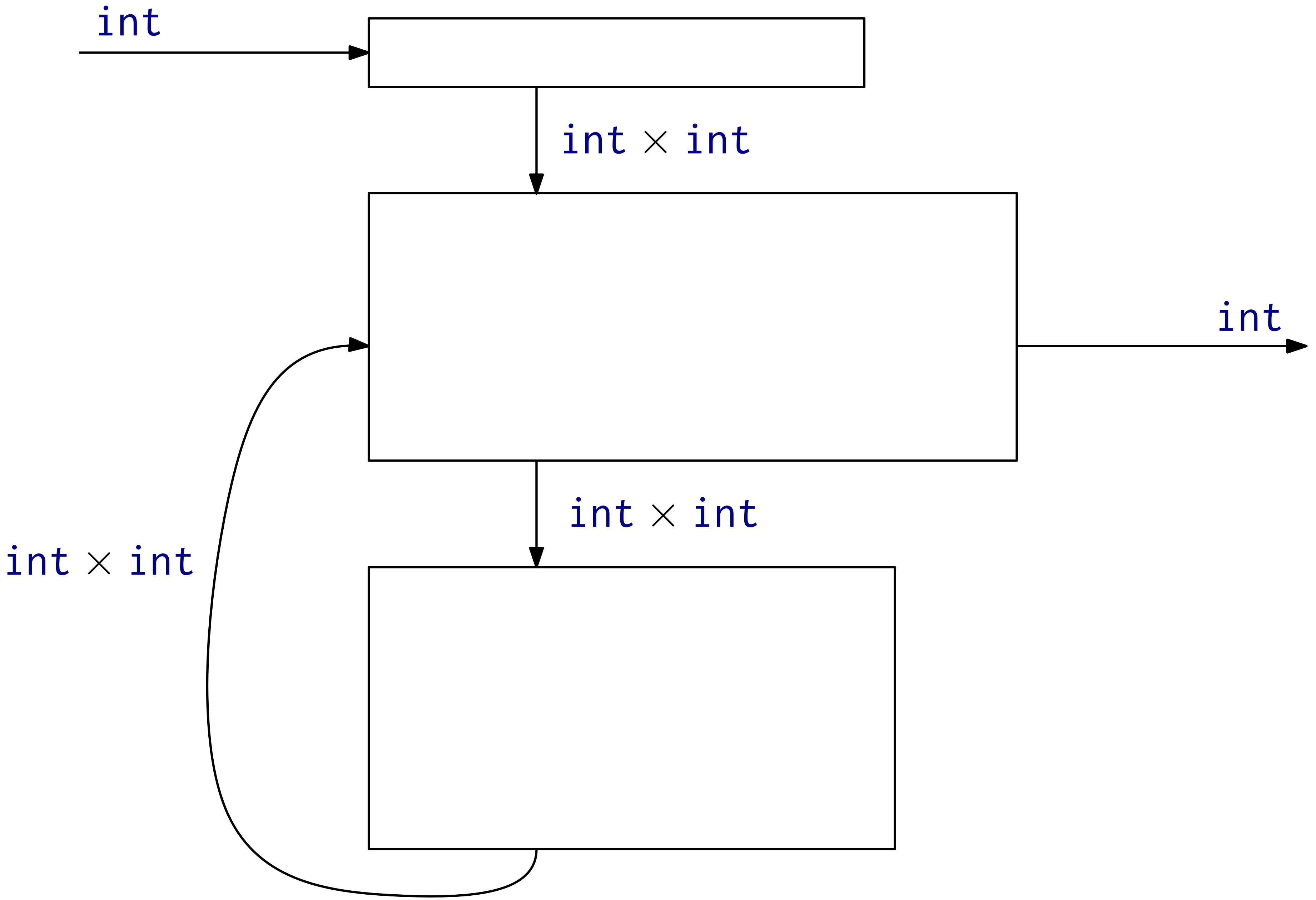
Blocks

$$
\begin{aligned}
&\mathtt{fac}(x : \mathtt{int}) = \mathtt{loop}(x, 1)\\[1em]
&\mathtt{loop}(x : \mathtt{int} \times \mathtt{int}) =\\
&\quad \mathsf{let}\ (n, acc) = x\ \mathsf{in}\\
&\quad \mathsf{let}\ b = eq(n, 1)\ \mathsf{in}\\
&\quad \mathsf{case}\ b\ \mathsf{of}\ \mathtt{inl}(\_) \Rightarrow \mathtt{ret}(acc)\\
&\qquad\qquad\quad ;\ \mathtt{inr}(\_) \Rightarrow \mathtt{body}(n, acc)\\[1em]
&\mathtt{body}(p : \mathtt{int} \times \mathtt{int}) =\\
&\quad \mathsf{let}\ (n, acc) = p\ \mathsf{in}\\
&\quad \mathsf{let}\ n' = sub(n, 1)\ \mathsf{in}\\
&\quad \mathsf{let}\ acc' = mul(acc, n)\ \mathsf{in}\\
&\quad \mathtt{loop}(n', acc')
\end{aligned}
$$

Control-Flow Graphs



Control-Flow Graphs



Operational Semantics

The execution of programs is a series of jumps:

- To begin execution, one jumps with some argument $v : A$ to some block:

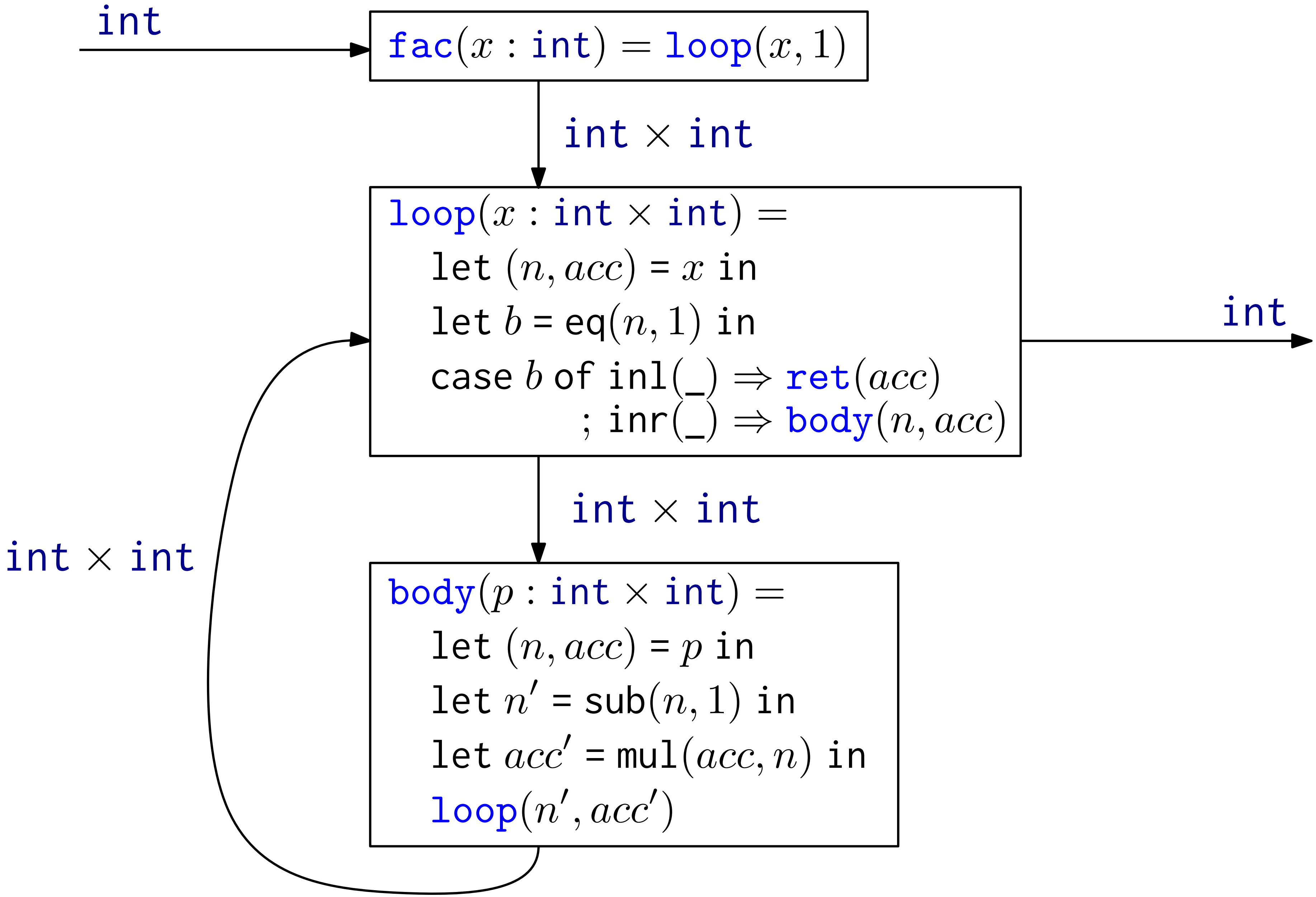
$$\texttt{label}(x : A) = body$$

- This will cause the $body$ to be evaluated.
- Evaluating $body$ ends with a jump to some other block.

Note

- There is no need for a call stack (or other side-effects).
- Effectful primitive operations may be added, if desired.

Operational Semantics



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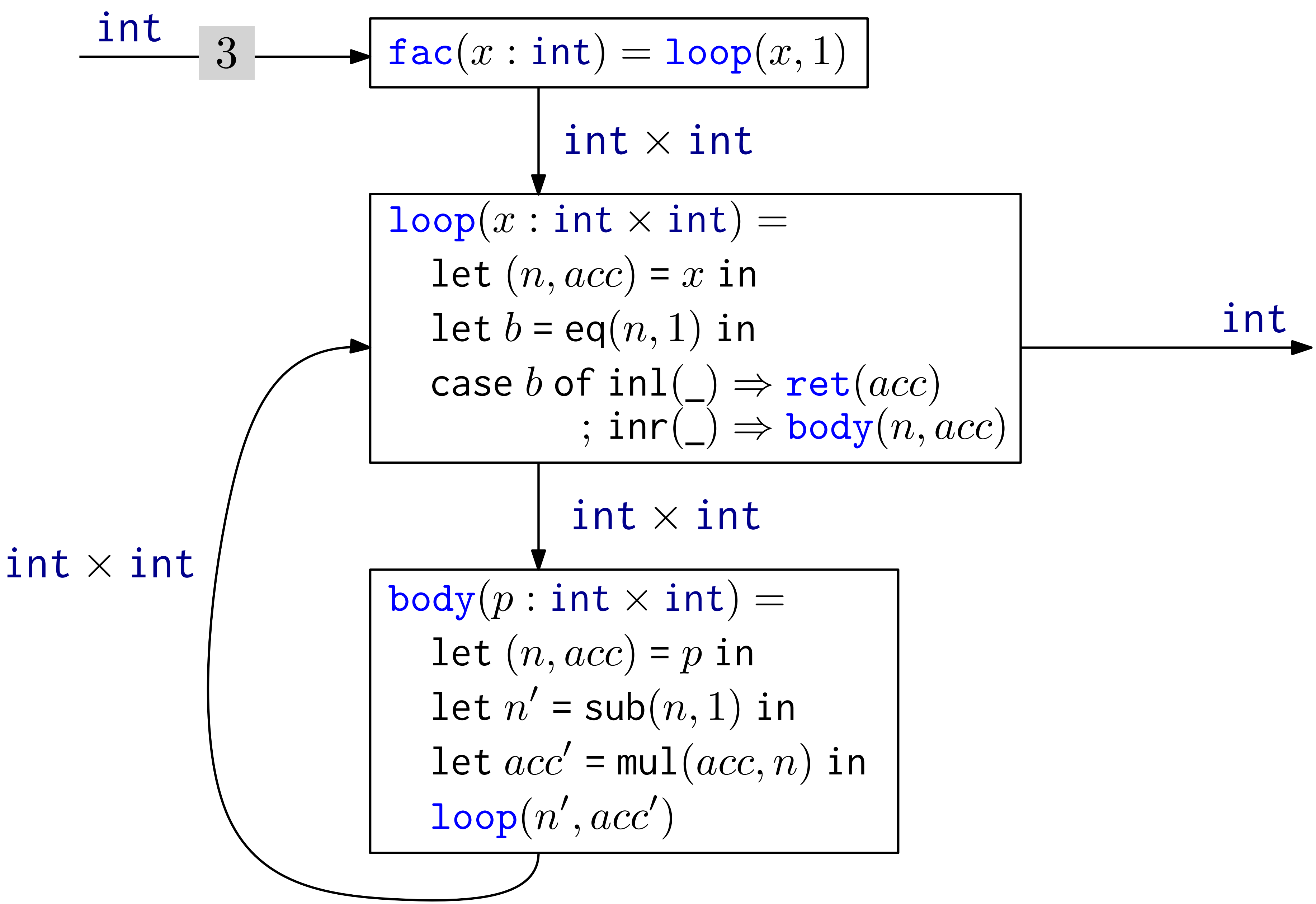

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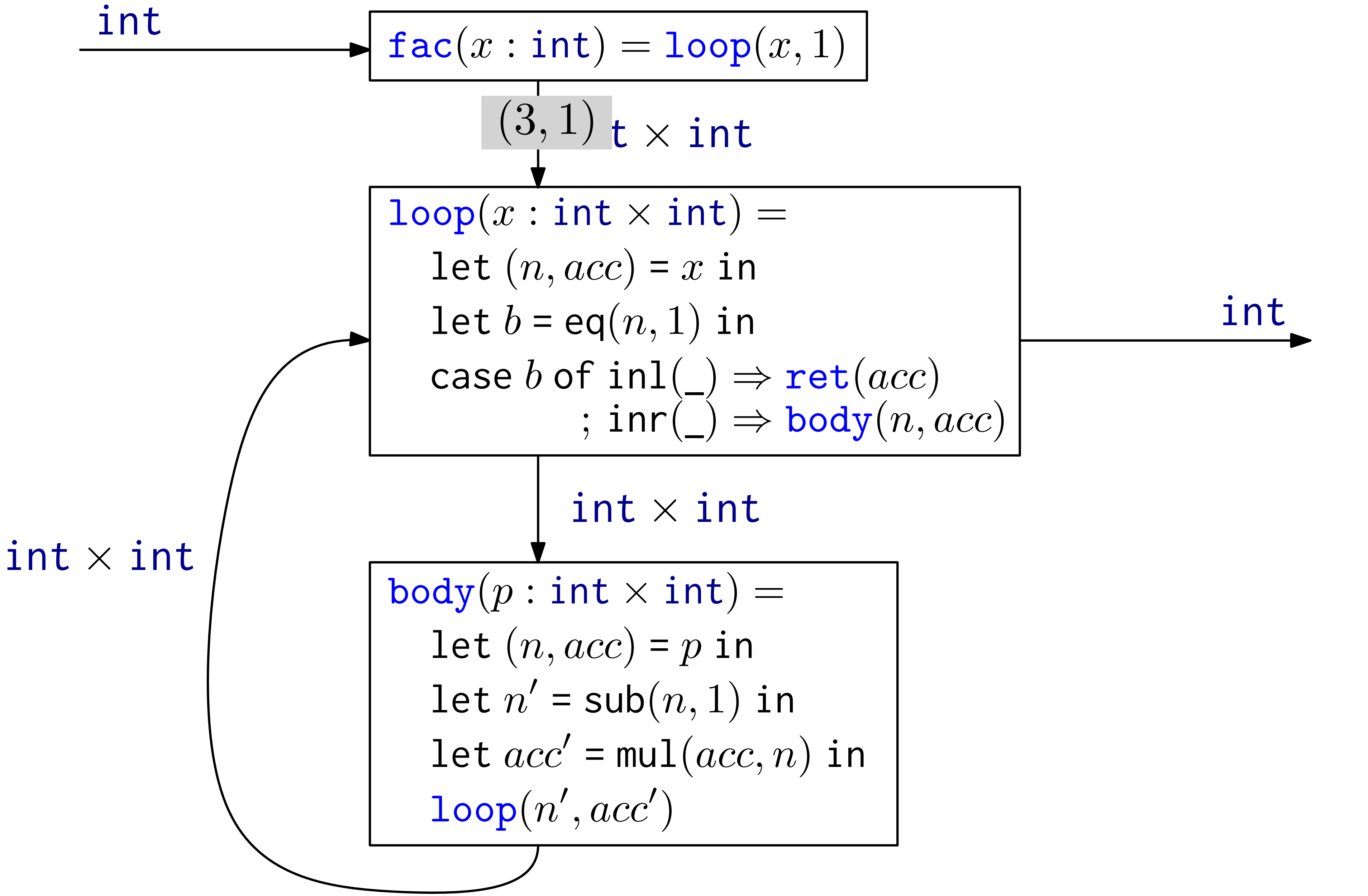

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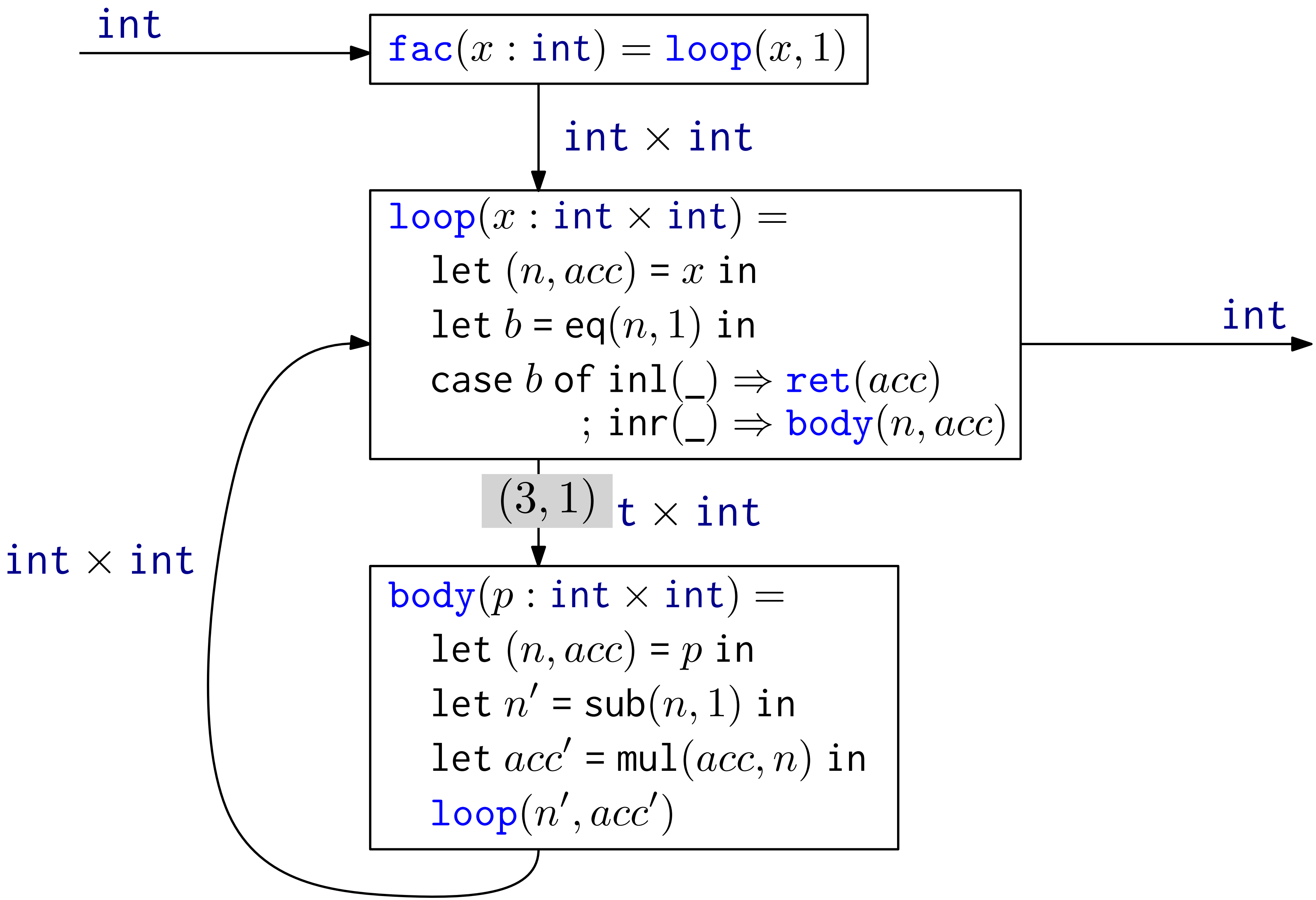

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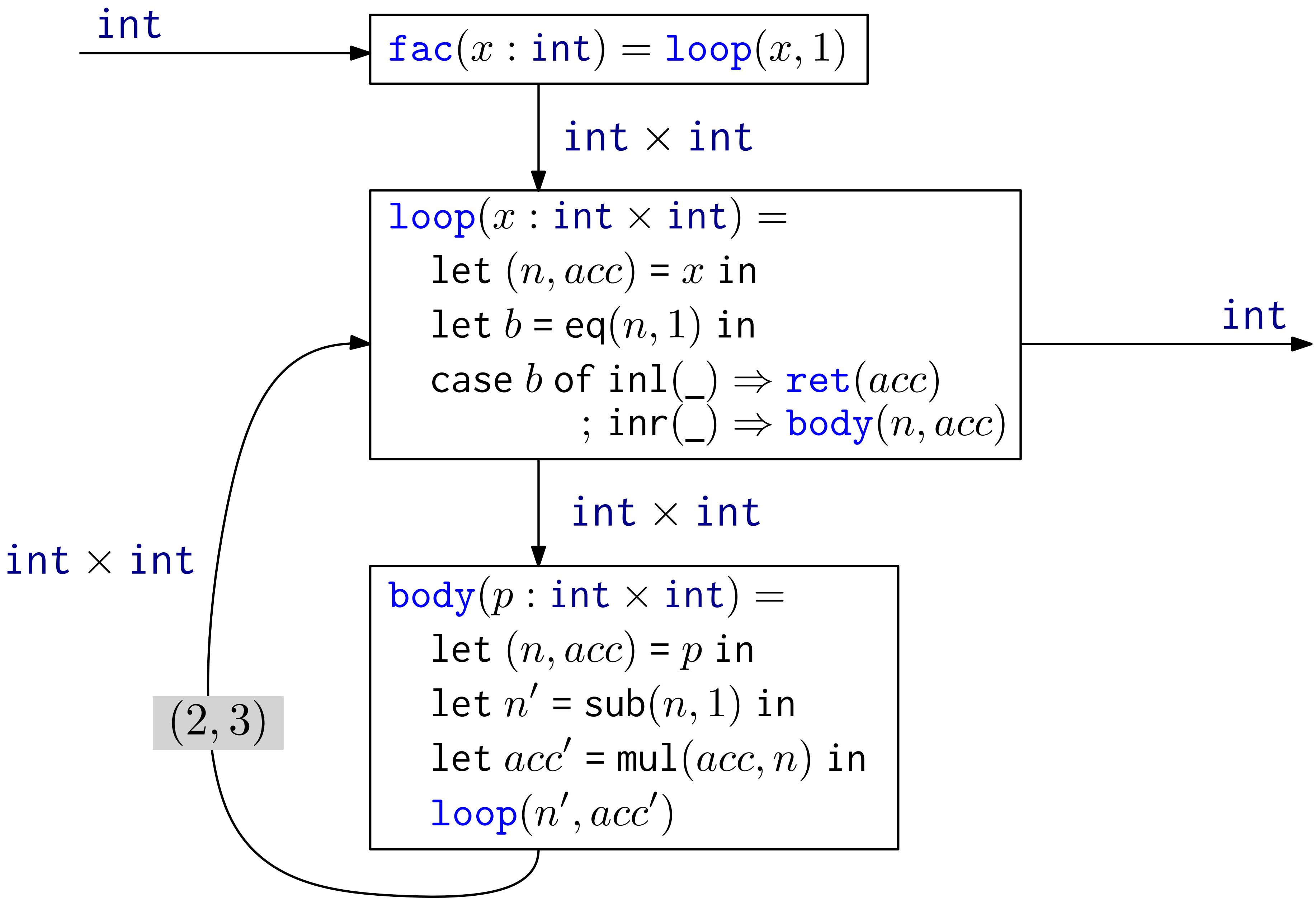

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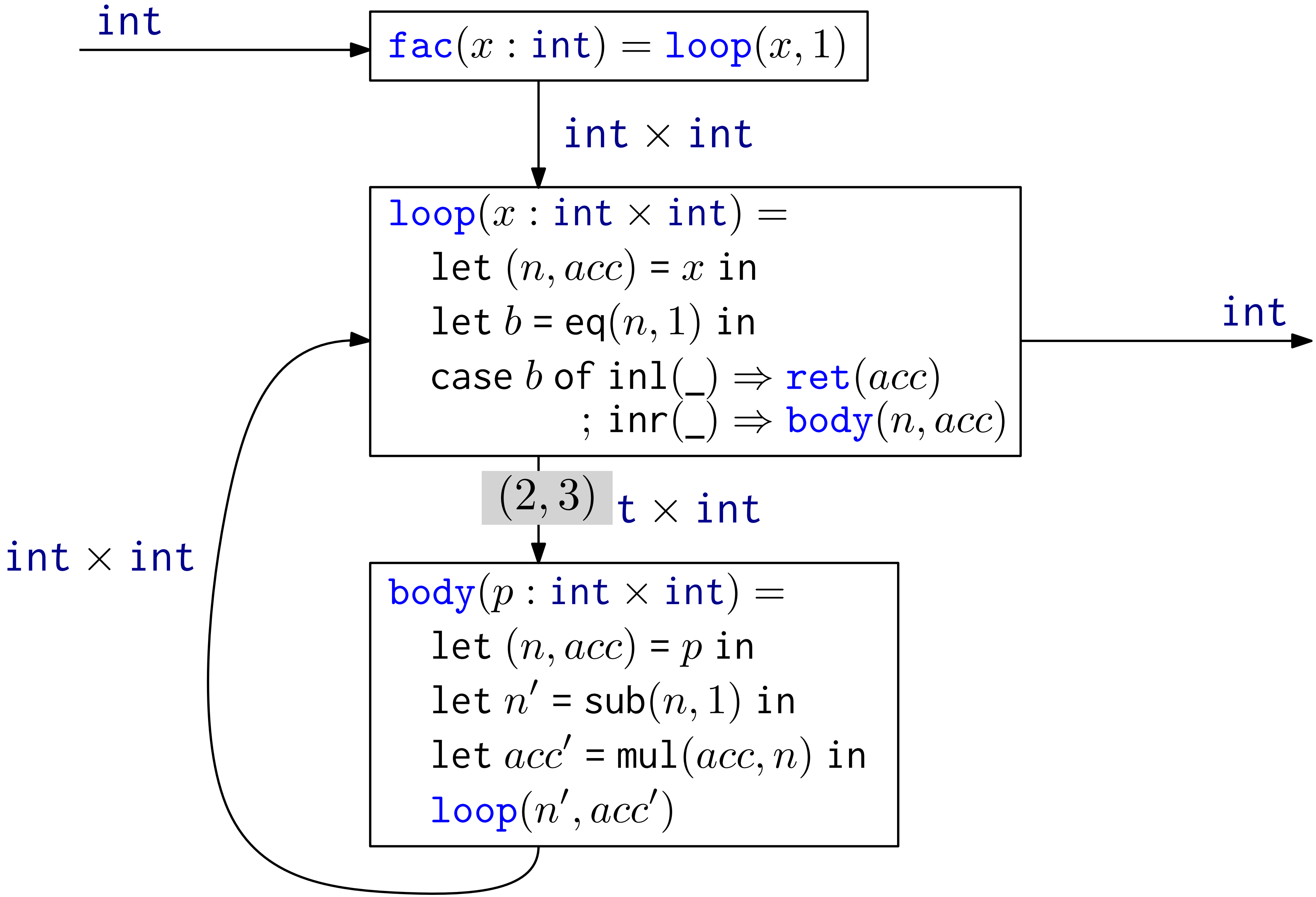

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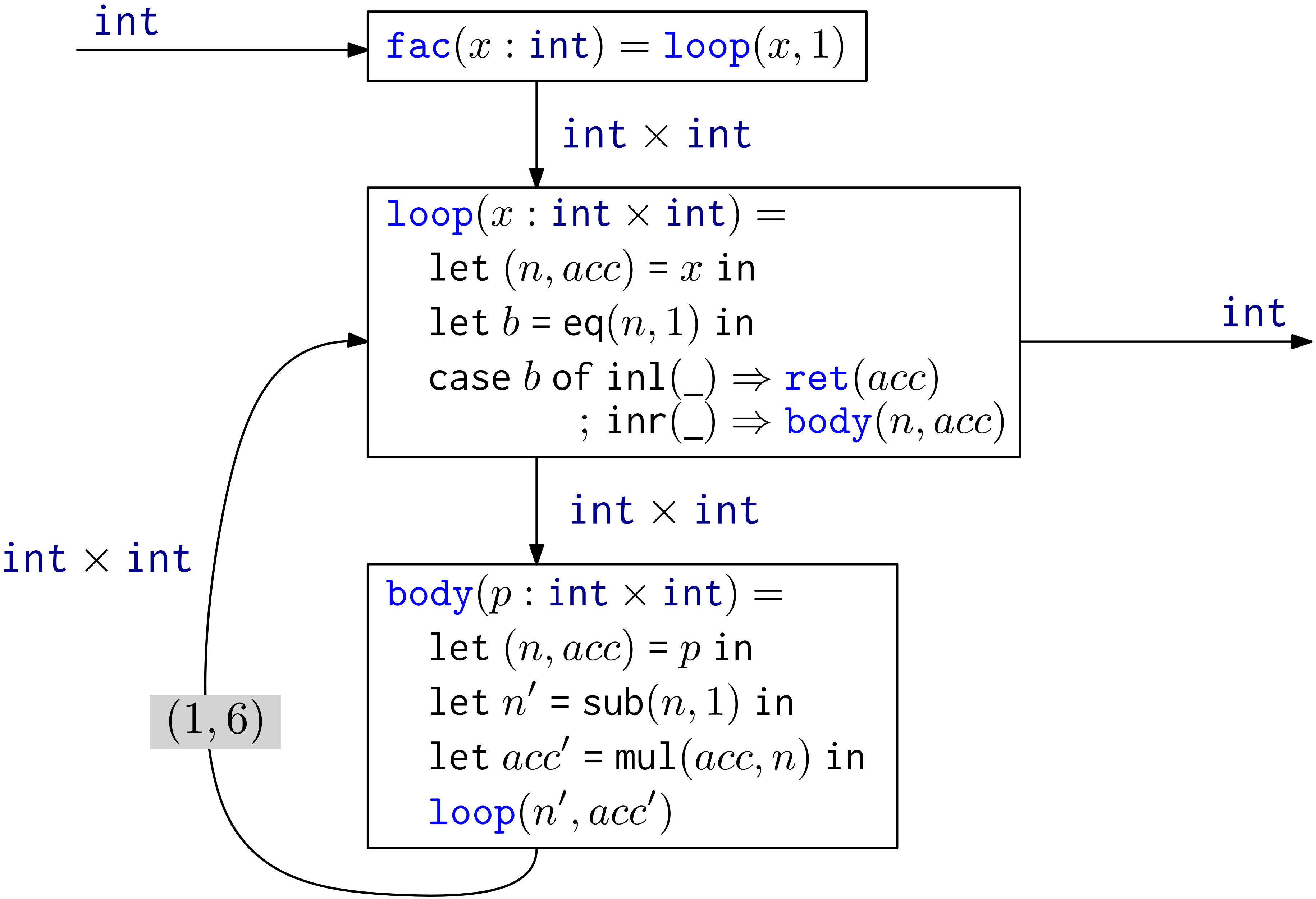

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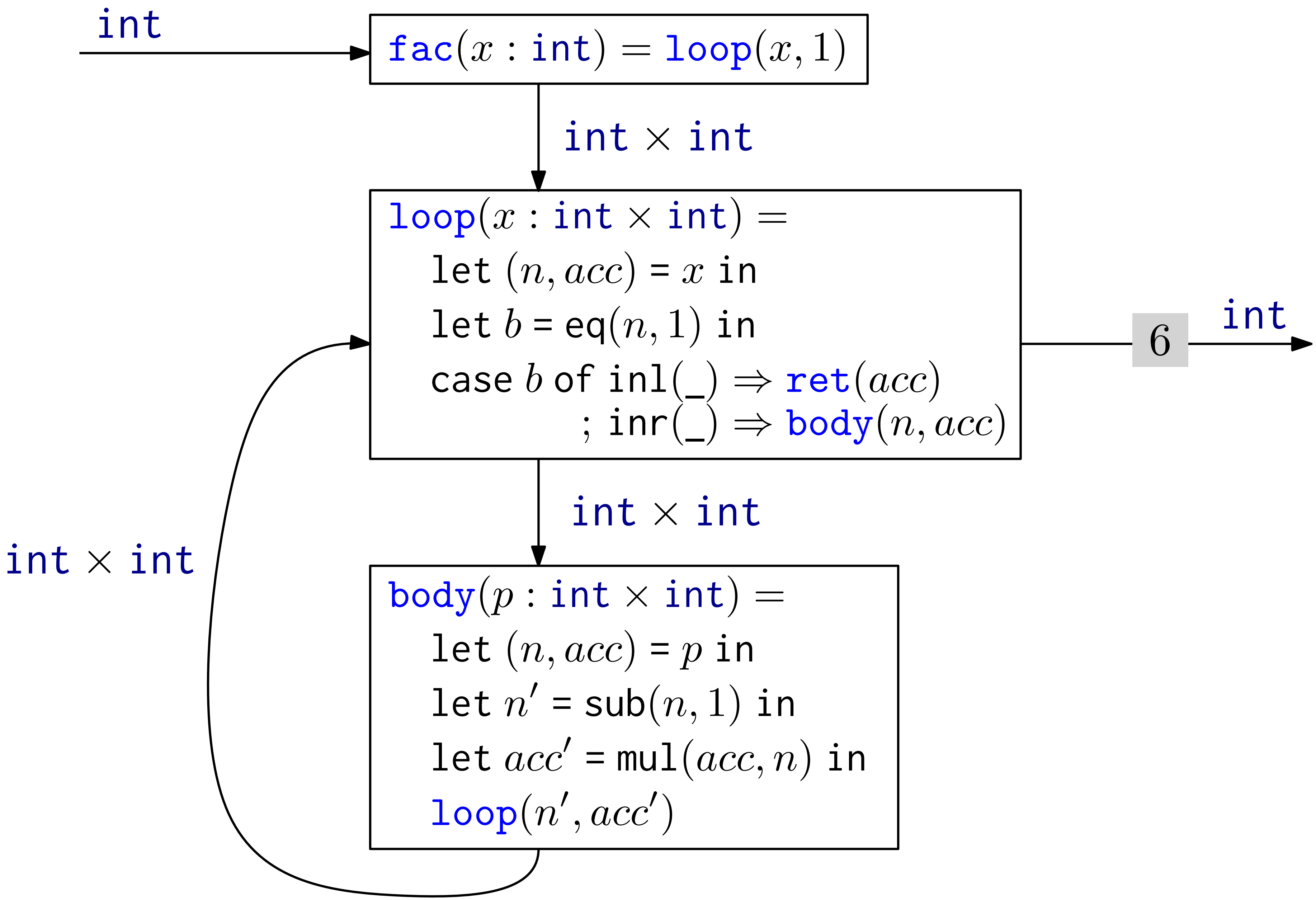
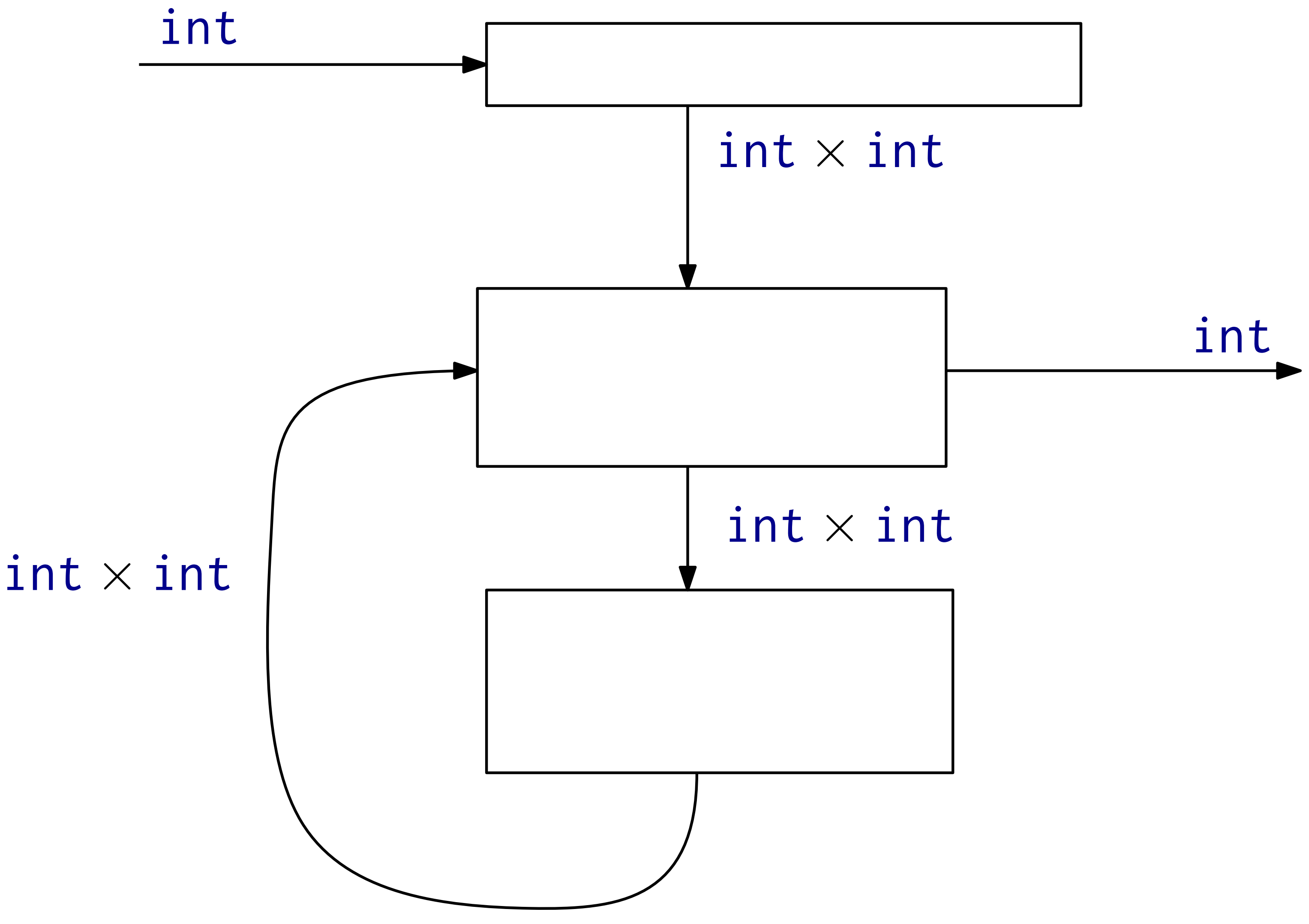


Program Fragment



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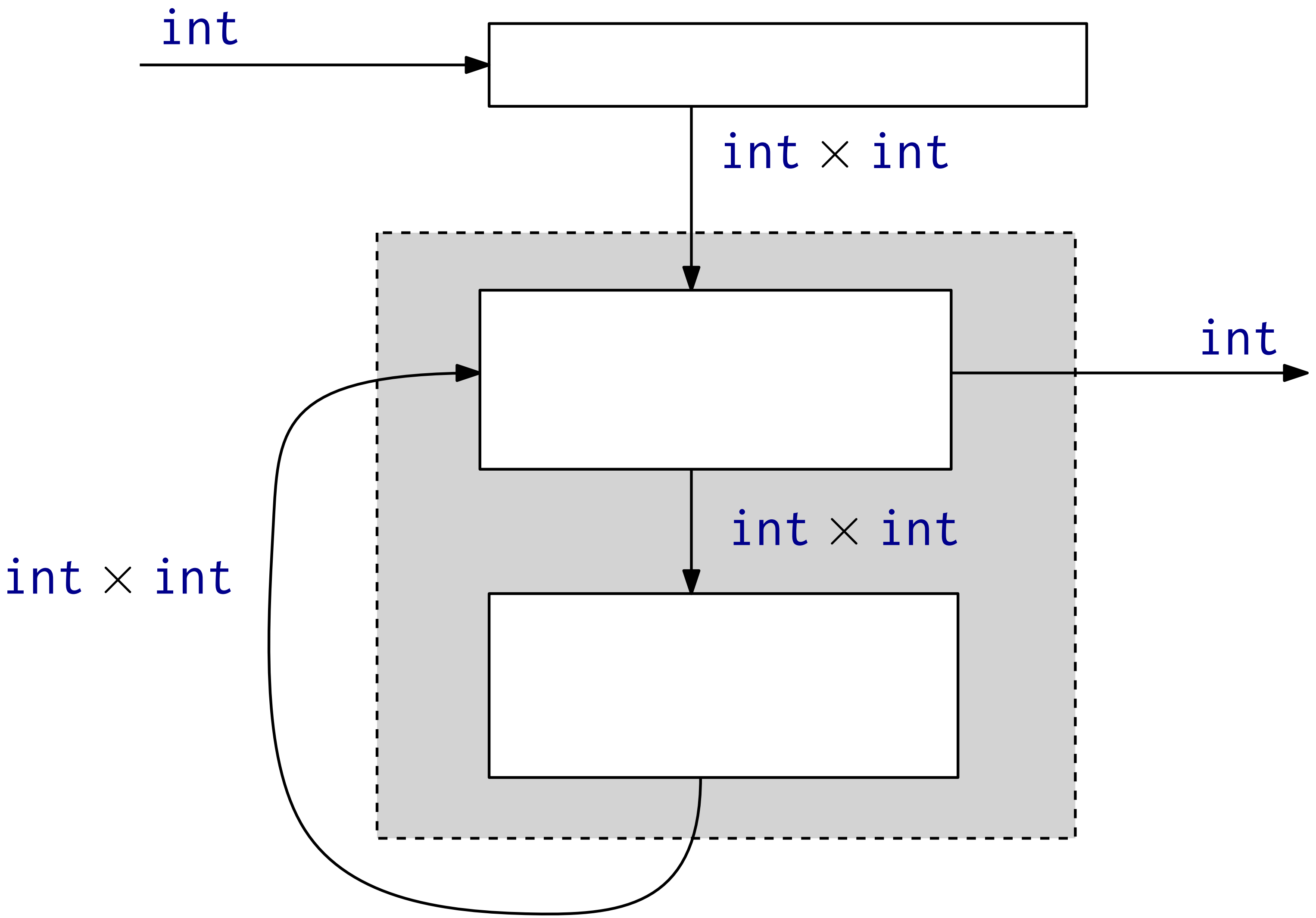

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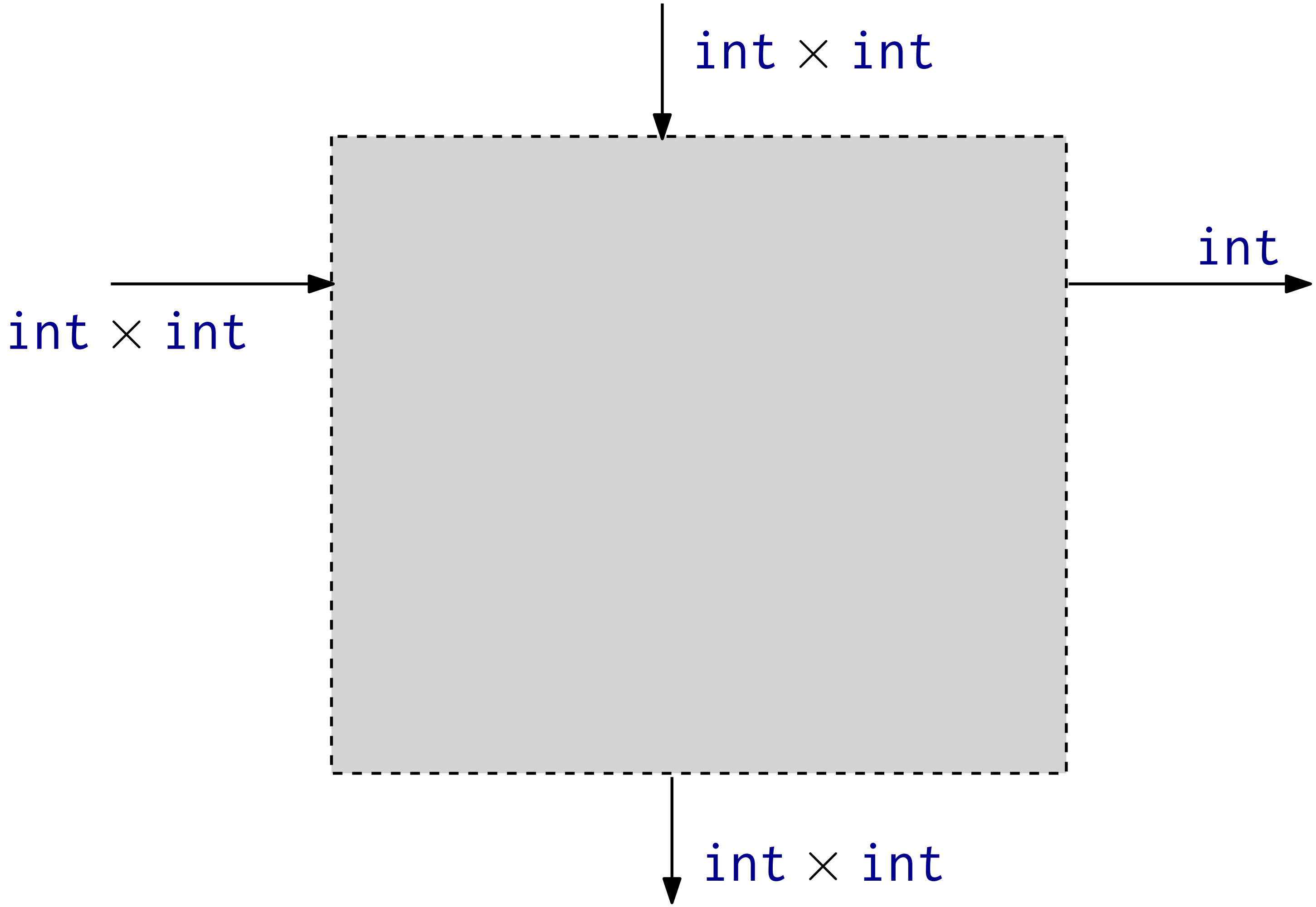
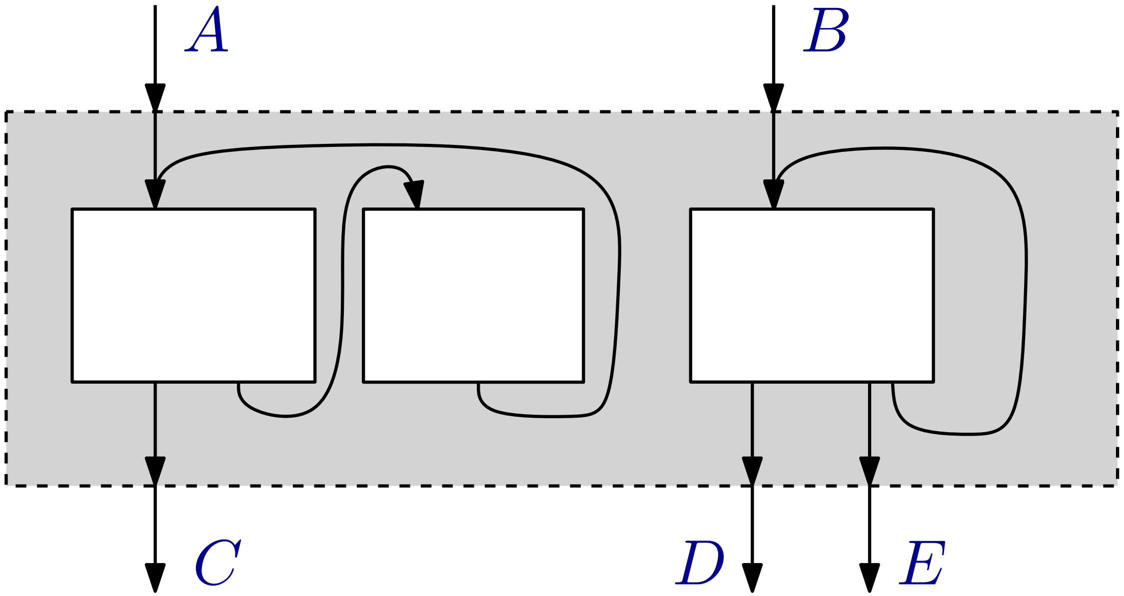


Program Fragment

A **program fragment** is a set of blocks (with pairwise distinct labels) together with

- a list of entry labels (each must be defined in a block),
- a list of exit labels (must not be defined in a block).

Graphical Notation



(i.e. control-flow graph with fixed entry- and exit-edges)

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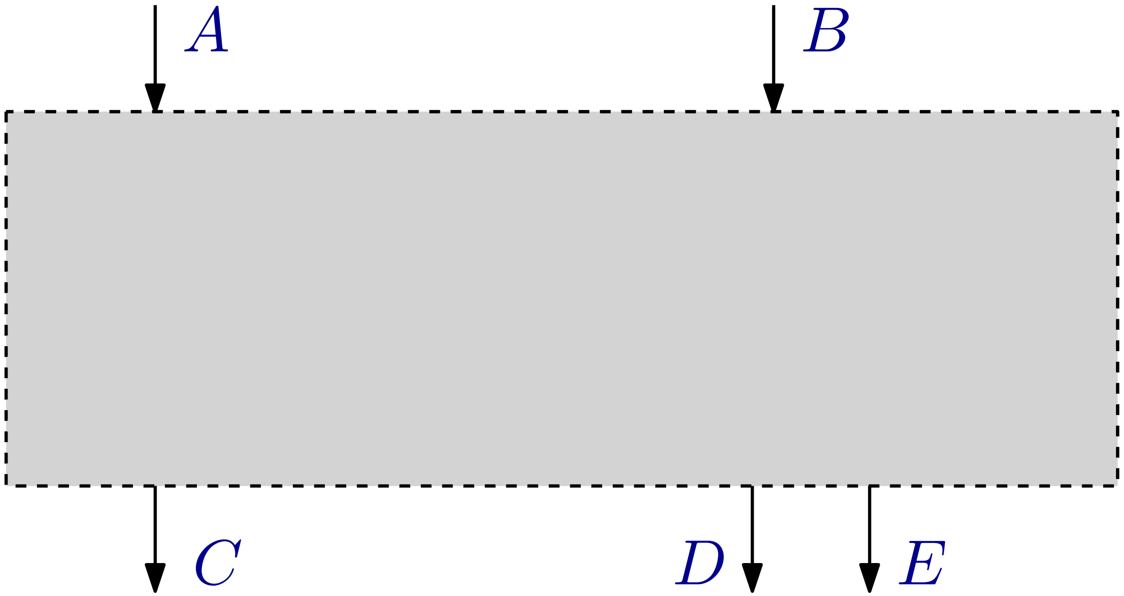

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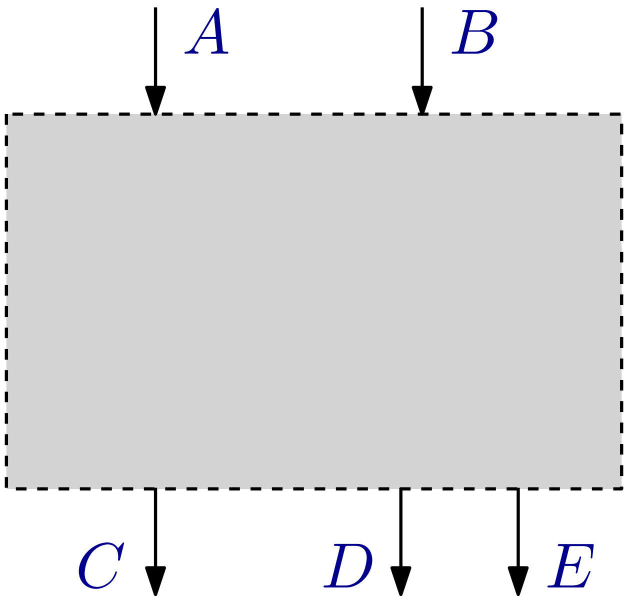
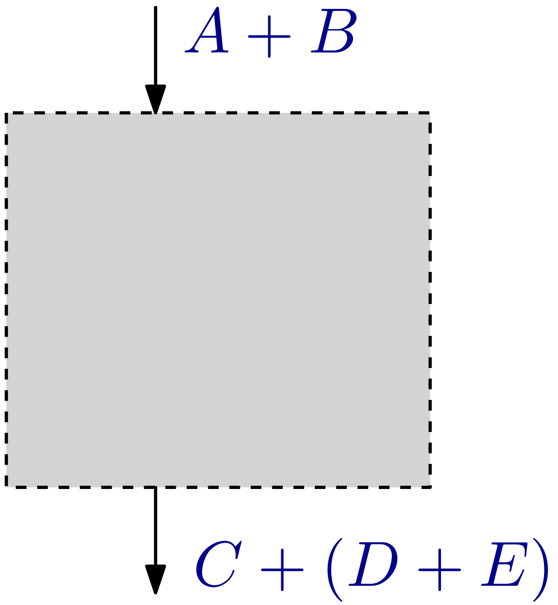
Graphical Notation

In the graphical notation, we omit trivial blocks (e.g. for associativity) and use types to disambiguate.

Example: implicit conversion



$\Longleftrightarrow$



Fragments with Free Variables

We will work with fragments that contain free value variables.

Example program with a free variable $z{:}\mathtt{int}$.

$$\mathtt{sum}(x : \mathtt{int}) = \mathtt{loop}(x, 1)$$

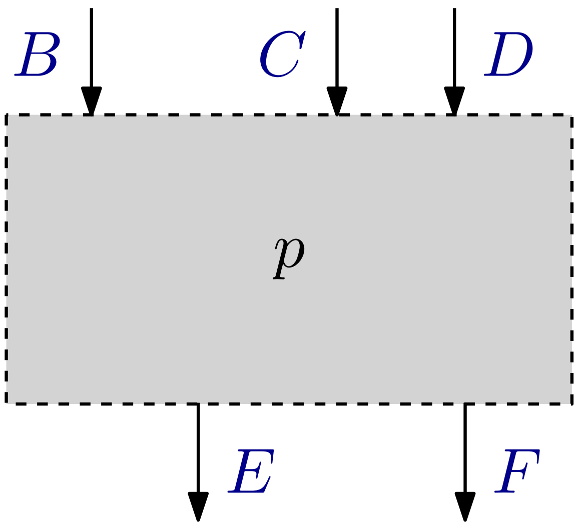
$$\begin{aligned}
&\mathtt{loop}(x : \mathtt{int} \times \mathtt{int}) = \\
&\quad \mathtt{let}\ (n, acc) = x\ \mathtt{in} \\
&\quad \mathtt{let}\ b = eq(n, 1)\ \mathtt{in} \\
&\quad \mathsf{case}\ b\ \mathsf{of}\ \mathtt{inl}(_) \Rightarrow \mathtt{ret}(acc) \\
&\qquad\qquad\quad ;\ \mathtt{inr}(_) \Rightarrow \mathtt{body}(n, acc)
\end{aligned}$$

$$\begin{aligned}
&\mathtt{body}(p : \mathtt{int} \times \mathtt{int}) = \\
&\quad \mathtt{let}\ (n, acc) = p\ \mathtt{in} \\
&\quad \mathtt{let}\ n' = \mathsf{sub}(n, 1)\ \mathtt{in} \\
&\quad \mathtt{let}\ acc' = \mathsf{add}(acc, z)\ \mathtt{in} \\
&\quad \mathtt{loop}(n', acc')
\end{aligned}$$

Graphical Notation — Box

Any program fragment p with a free variable $z{:}A$ can be transformed into a fragment, where z is not free, but is passed around as an argument.

Add a new first parameter of type A to all the blocks in p.



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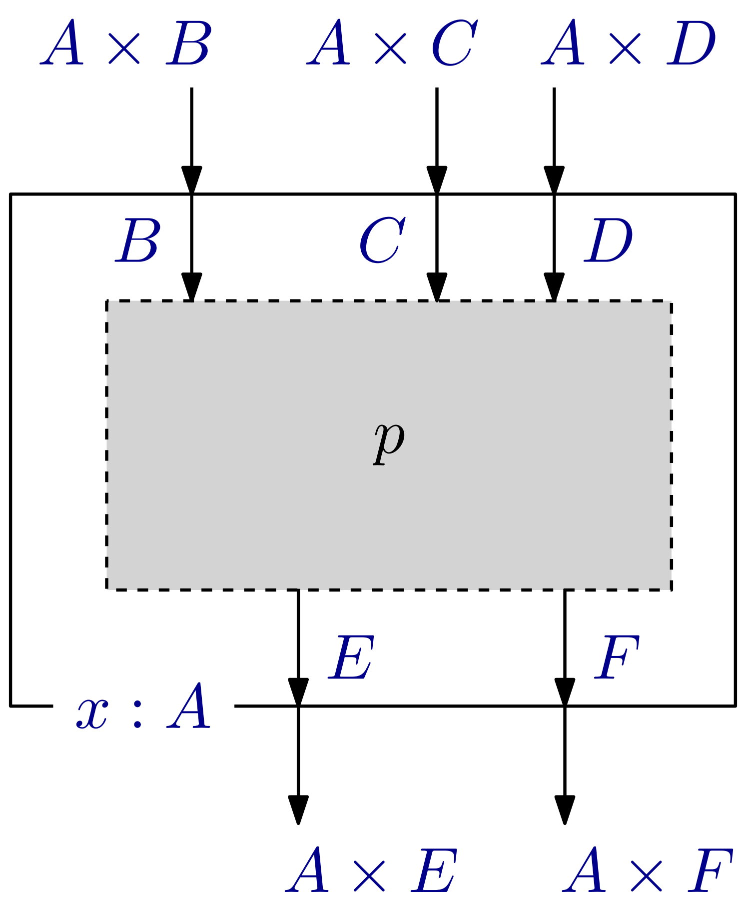

Graphical Notation — Box

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Example:

$$
\begin{array}{l}
\texttt{label1}(y : B) = \\
\quad \texttt{let}\ b\ \texttt{=}\ \texttt{...}\ \texttt{in} \\
\quad \texttt{case}\ b\ \texttt{of}\ \texttt{inl}(\_) \Rightarrow \texttt{label2}(v) \\
\qquad\qquad\quad ;\ \texttt{inr}(\_) \Rightarrow \texttt{label3}(w)
\end{array}
\quad\Longrightarrow\quad
\begin{array}{l}
\texttt{label1}(z : A \times B) = \\
\quad \texttt{let}\ (x, y)\ \texttt{=}\ z\ \texttt{in} \\
\quad \texttt{let}\ b\ \texttt{=}\ \texttt{...}\ \texttt{in} \\
\quad \texttt{case}\ b\ \texttt{of}\ \texttt{inl}(\_) \Rightarrow \texttt{label2}(x, v) \\
\qquad\qquad\quad ;\ \texttt{inr}(\_) \Rightarrow \texttt{label3}(x, w)
\end{array}
$$

Organising Low-Level Programs

Constructions from Interaction Semantics

Int Construction

Free compact closed completion of a traced symmetric monoidal category [Joyal, Street, Verity 1994].

Captures the core of many constructions, in particular:

- Game Semantics [Abramsky, Jagadeesan, Malacaria 2001]
- Geometry of Interaction [Haghverdi, Scott 2004]

Int Construction

Construction of the **integers** from the **natural numbers**.

Represent integers by pairs $(i^-, i^+) \in \mathbb{N} \times \mathbb{N}$.

$$
\begin{aligned}
(x^-, x^+) \leq (y^-, y^+) \quad &\iff \quad x^+ - x^- \leq y^+ - y^- \\
&\iff \quad x^+ + y^- \leq x^- + y^+
\end{aligned}
$$

Int Construction

Construction of the **integers** from the **natural numbers**.

Represent integers by pairs $(i^-, i^+) \in \mathbb{N} \times \mathbb{N}$.

$$(x^-, x^+) \leq (y^-, y^+) \iff x^+ - x^- \leq y^+ - y^-$$
$$\iff x^+ + y^- \leq x^- + y^+$$

Generalise from the **natural numbers** $(\mathbb{N}, +, 0)$ to a **traced symmetric monoidal category** $(\mathbb{C}, \oplus, 0)$.

Objects: pairs (X^-, X^+) of $\mathbb{C}$-objects

Morphisms:

$$(X^-, X^+) \to (Y^-, Y^+) \iff X^+ \oplus Y^- \to X^- \oplus Y^+ \text{ in } \mathbb{C}$$

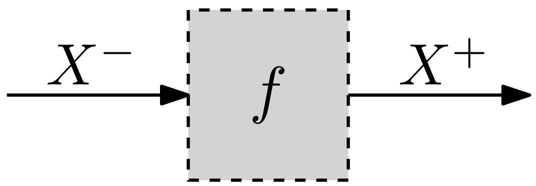
Int Construction for Low-Level Programs

An object (X^-, X^+) models the **interface** of interactive entity.

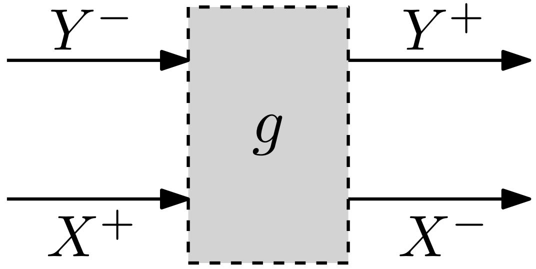
type of possible **answers** from entity

type of possible **questions** to entity

Implementation of interface X:

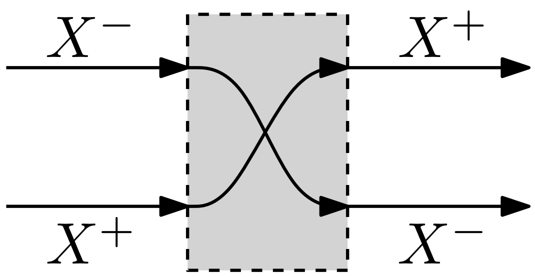


A morphism $X \longrightarrow Y$ is an implementation of interface Y with possible queries to X

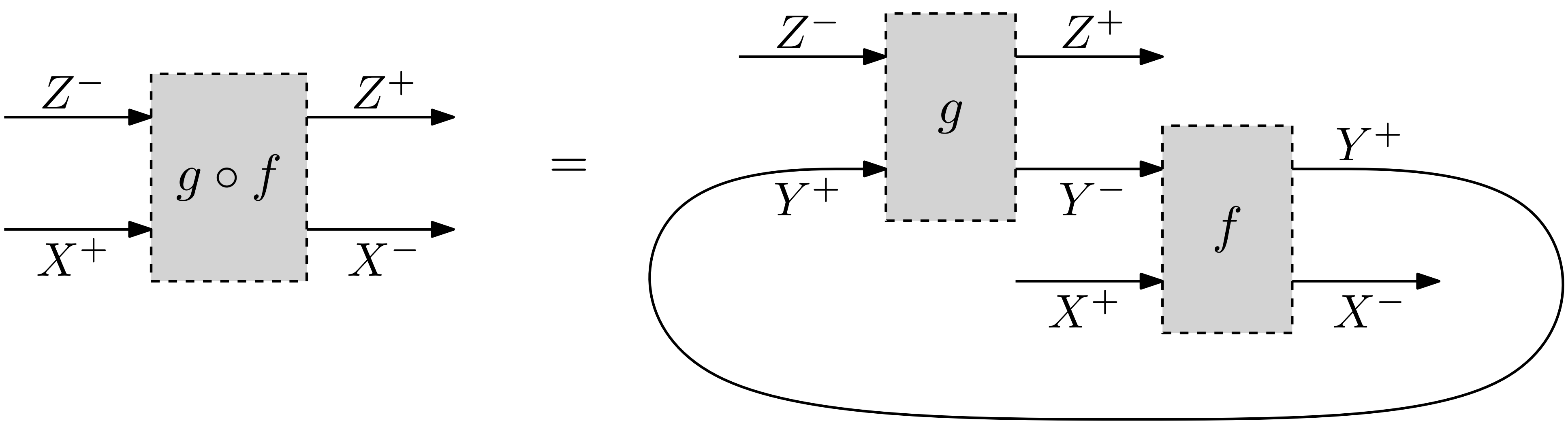


Int Construction for Low-Level Programs

Identity



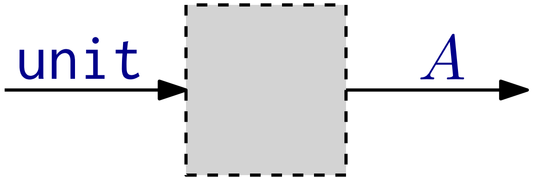
Composition of $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$.



Types of Interaction: $[A]$

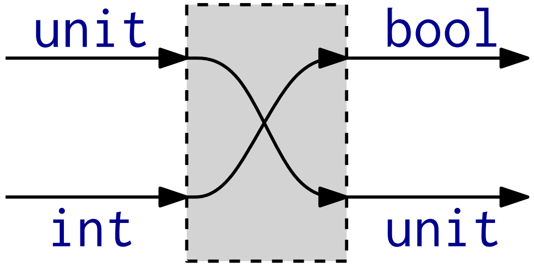
Base $[A]$

- questions: $[A]^- = \texttt{unit}$
- answers: $[A]^+ = A$



Example:

$$\texttt{even}\colon [\texttt{int}] \longrightarrow [\texttt{bool}]$$



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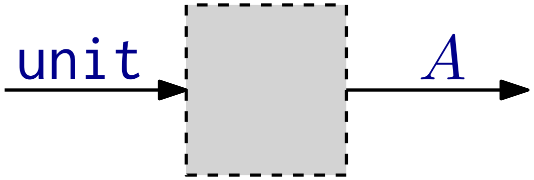

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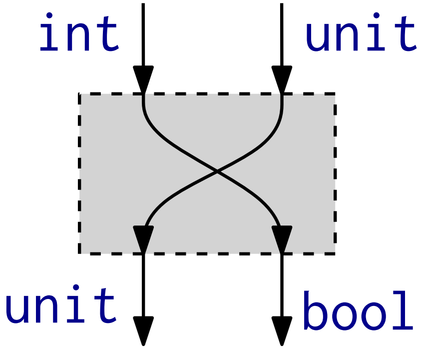
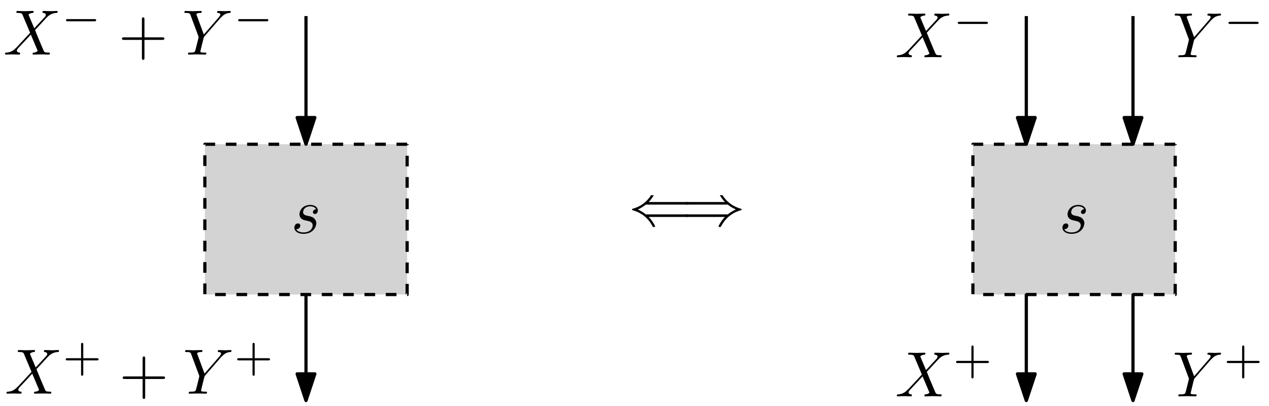


Types of Interaction: $X \otimes Y$

Pairs $X \otimes Y$

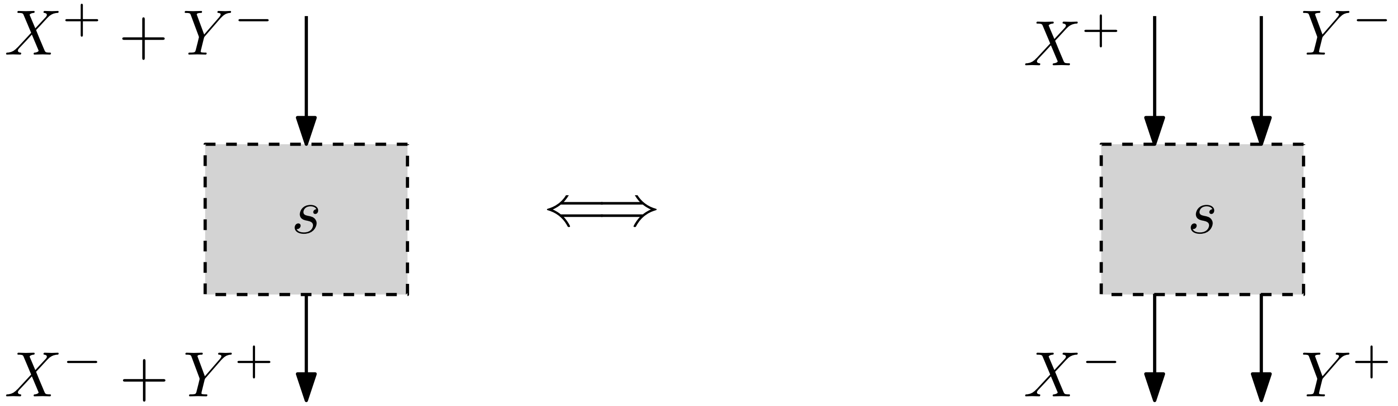
- questions: $(X \otimes Y)^- = X^- + Y^-$
- answers: $(X \otimes Y)^+ = X^+ + Y^+$



Types of Interaction: $X \multimap Y$

Interactive Functions $X \multimap Y$

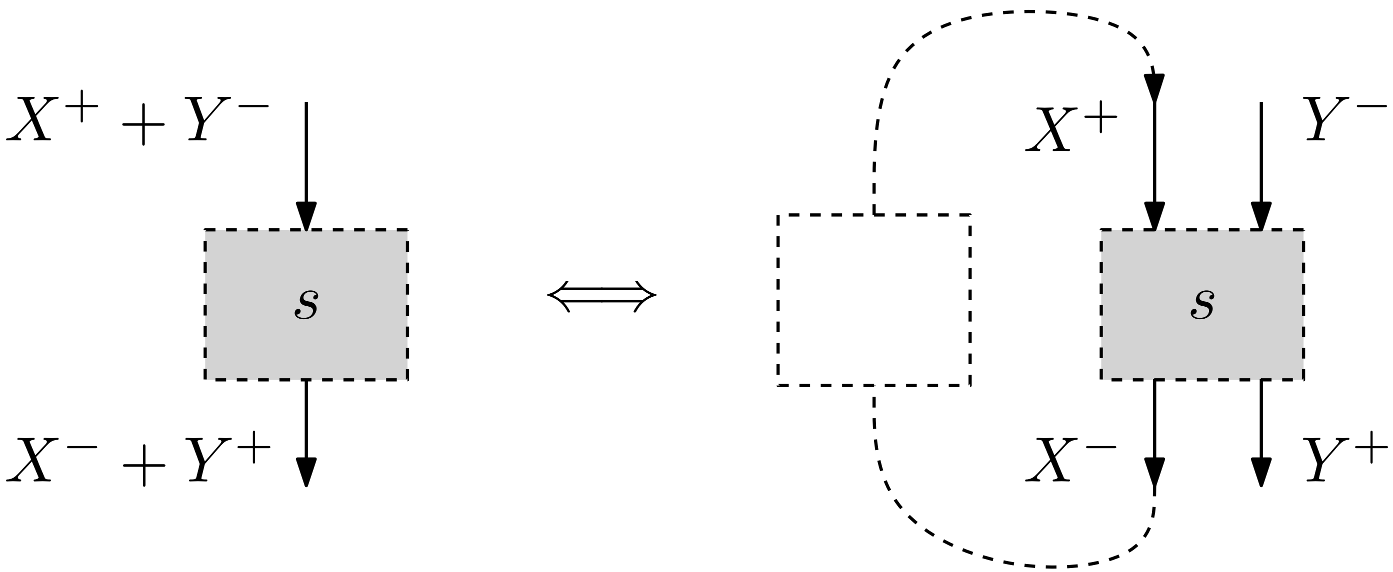
- questions: $(X \multimap Y)^- = X^+ + Y^-$
- answers: $(X \multimap Y)^+ = X^- + Y^+$



Types of Interaction: $X \multimap Y$

Interactive Functions $X \multimap Y$

- questions: $(X \multimap Y)^- = X^+ + Y^-$
- answers: $(X \multimap Y)^+ = X^- + Y^+$



Types of Interaction: $X \multimap Y$

Problem: There is no addition function

$$\mathsf{add}\colon [\mathtt{int}] \multimap [\mathtt{int}] \multimap [\mathtt{int}].$$

We cannot remember values between requests.

Types of Interaction: $A \cdot X$

Subexponential $A \cdot X$

- questions: $(A \cdot X)^- = A \times X^-$
- answers: $(A \cdot X)^+ = A \times X^+$

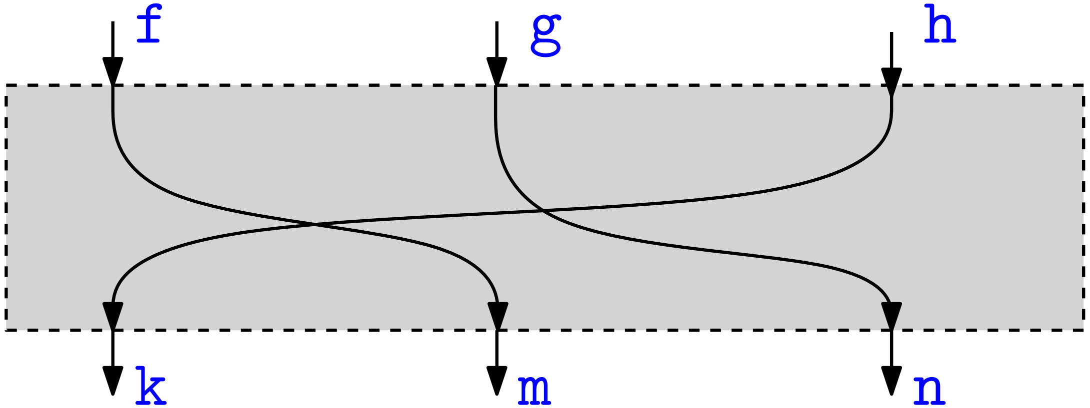
Callee-Save-Invariant

- The value of type A is returned unchanged.
- The answer may not depend on the value of type A.

Types of Interaction: $A \cdot X$

Example

$[A]$	$\multimap$	$A \cdot [B]$	$\multimap$	$[A \times B]$
				$()$
$()$				
a				
		$(a, ())$		
		(a, b)		
				(a, b)



$\mathtt{f}(a : A) = \mathtt{m}(a, ())$

$\mathtt{g}(x : A \times B) = \mathtt{n}(x)$

$\mathtt{h}(x : \mathtt{unit}) = \mathtt{k}(x)$

Types of Interaction: $A \cdot X$

Example

$$\underbrace{(\texttt{unit} + A)}_{S} \cdot ([\texttt{bool}] \multimap [A]) \quad \multimap \quad [A \times A]$$

$()$

$(\texttt{inl}(), ())$

$(\texttt{inl}(), ())$

$(\texttt{inl}(), \texttt{false})$

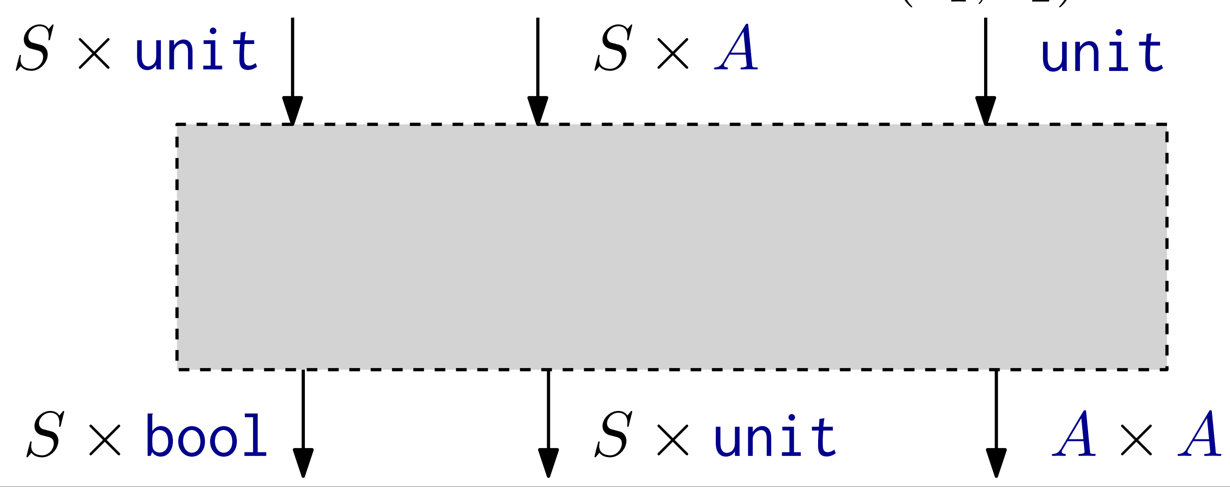
$(\texttt{inl}(), a_1)$

$(\texttt{inr}(a_1), ())$

$(\texttt{inr}(a_1), \texttt{true})$

$(\texttt{inr}(a_1), a_2)$

(a_1, a_2)



Types of Interaction: $A \cdot X$

Examples

$$\texttt{unit} \cdot X \multimap X$$

$$A \cdot (B \cdot X) \multimap (A \times B) \cdot X$$

$$(A + B) \cdot X \multimap (A \cdot X) \otimes (B \cdot X)$$

Types of Interaction: $!X$

Exponential $!X$

$$!X := \mathtt{nat} \cdot X$$

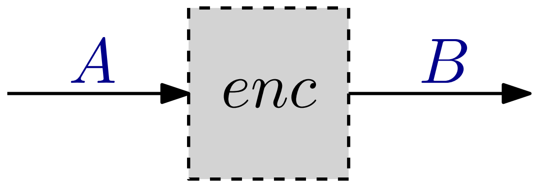
where

$$\begin{aligned} \mathtt{type}\ \mathtt{nat} &= \mathsf{Zero}\ \mathsf{of}\ \mathtt{unit} \\ &\mid \mathsf{Succ}\ \mathsf{of}\ \mathtt{nat} \end{aligned}$$

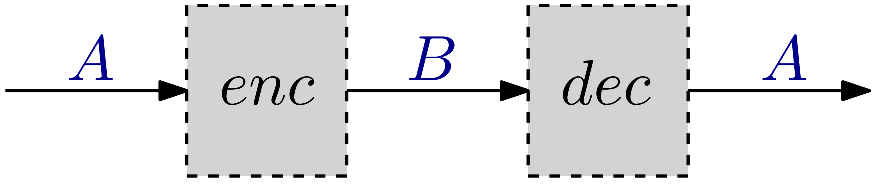
Game Semantics and Geometry of Interaction use this special case.

Retraction $A \lhd B$

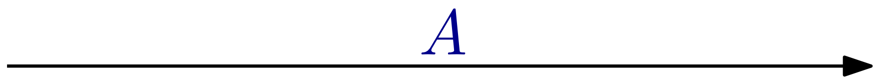
Write $A \lhd B$ if there *exist* two program fragments



such that



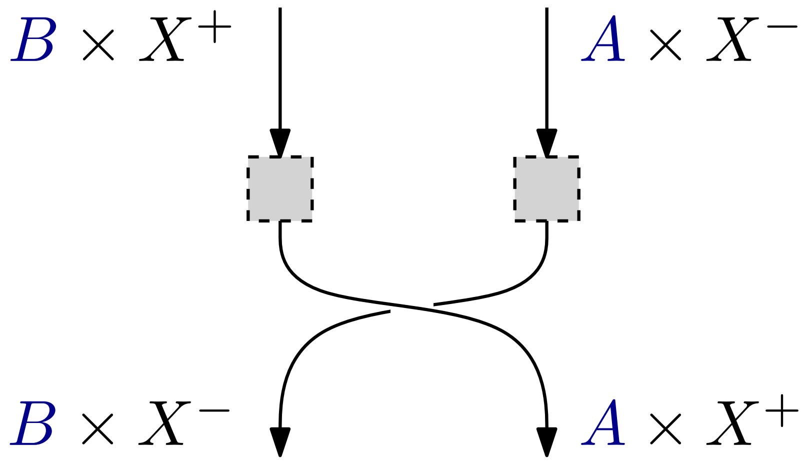
behaves like



Types of Interaction: $!X$

If $A \lhd B$, then we can define a map:

$$B \cdot X \multimap A \cdot X$$



Why not always use $!X$?

- $!X$ requires encoding into `nat`; the compiler for the low-level language would have to undo this for optimisation.
- Encoding details can become visible in some applications, e.g. Bounded Linear Logic [Girard, Scedrov, Scott 1992]

$$(A + B) \cdot X \multimap (A \cdot X) \otimes (B \cdot X)$$

$$!_{p+q} X \multimap !_p X \otimes !_q X$$

$$\begin{aligned} enc(\mathtt{inl}(v)) &= enc(v) \\ enc(\mathtt{inr}(w)) &= p + enc(w) \end{aligned}$$

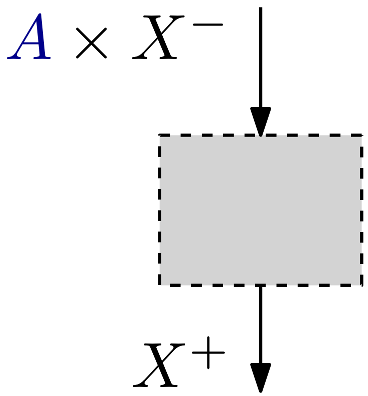
$$!_{2\cdot\max(p,q)} X \multimap !_p X \otimes !_q X$$

$$\begin{aligned} enc(\mathtt{inl}(v)) &= \langle 0, enc(v) \rangle \\ enc(\mathtt{inr}(w)) &= \langle 1, enc(w) \rangle \end{aligned}$$

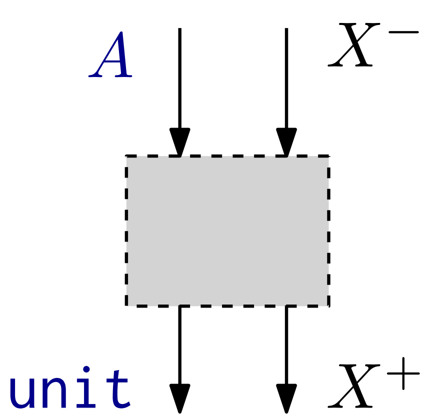
Types of Interaction: $A \to X$

Value Passing $A \to X$

- questions: $(A \to X)^- = A \times X^-$
- answers: $(A \to X)^+ = X^+$



Compare: $[A] \multimap X$



Types of Interaction: $\forall\alpha \lhd A.\, X$ and $\exists\alpha \lhd A.\, X$

Value-Type Polymorphism $\forall\alpha \lhd A.\, X$

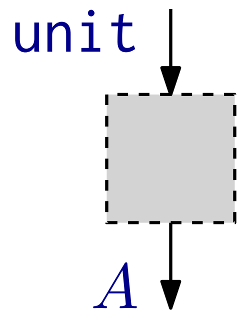
- questions: $(\forall\alpha \lhd A.\, X)^- = (\exists\alpha \lhd A.\, X)^- = X^-[A/\alpha]$
- answers: $(\forall\alpha \lhd A.\, X)^+ = (\exists\alpha \lhd A.\, X)^+ = X^+[A/\alpha]$

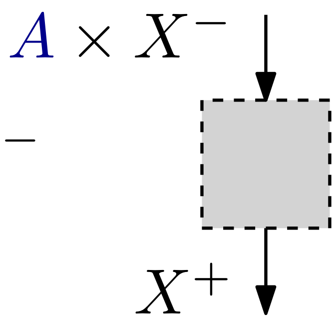
Special cases:

$$\forall\alpha.\, X := \forall\alpha \lhd \mathsf{nat}.\, X$$
$$\exists\alpha.\, X := \forall\alpha \lhd \mathsf{nat}.\, X$$

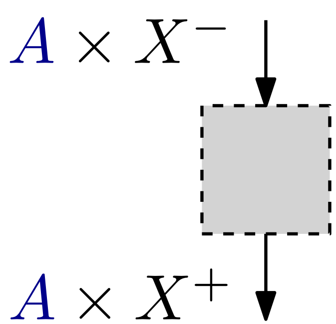
Summary

$$[A]^- = \texttt{unit}$$
$$[A]^+ = A$$



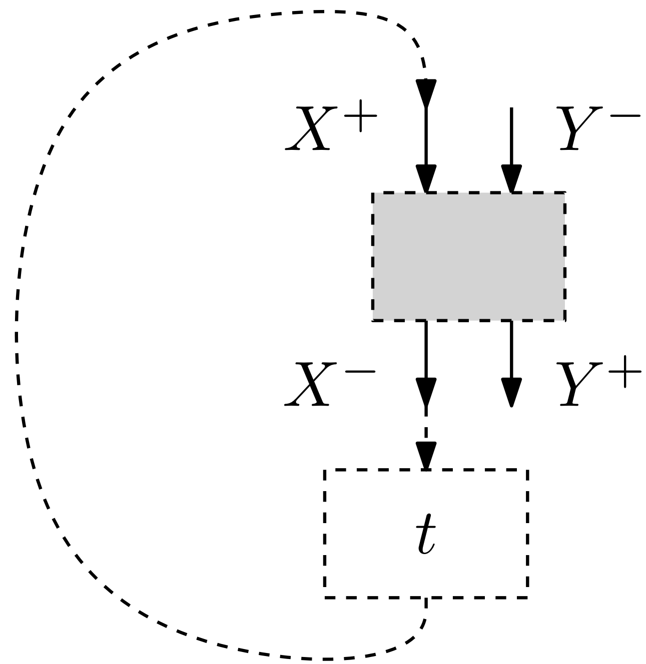


$$(A \to X)^- = A \times X^-$$
$$(A \to X)^+ = X^+$$



$$(A \cdot X)^- = A \times X^-$$
$$(A \cdot X)^+ = A \times X^+$$

$$(X \multimap Y)^- = X^+ + Y^-$$
$$(X \multimap Y)^+ = X^- + Y^+$$



$$(\forall\alpha.X)^- = X^-[\mathsf{nat}/\alpha]$$
$$(\forall\alpha.X)^+ = X^+[\mathsf{nat}/\alpha]$$