MINIMAL FROM CLASSICAL PROOFS

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Abstract. Let A be a formula without implications, and Γ consist of formulas containing disjunction and falsity only negatively and implication only positively. Orevkov (1968) and Nadathur (2000) proved that classical derivability of A from Γ implies intuitionistic derivability, by a transformation of derivations in sequent calculi. We give a new proof of this result (for minimal rather than intuitionistic logic), where the input data are natural deduction proofs in long normal form (given as proof terms via the Curry-Howard correspondence) involving stability axioms for relations; the proof gives a quadratic algorithm to remove the stability axioms. This can be of interest for computational uses of classical proofs. Keywords: Minimal logic, stability axioms, Glivenko-style theorems, Orevkov, intuitionistic logic.

It is well-known that in certain situations classical provability implies constructive provability. Glivenko proved in [5] that every negated propositional formula provable in classical logic is provable intuitionistically. Another famous "Glivenko-style" result is Barr's theorem [1], which deals with geometric formulas $\exists_{\vec{x}}(B_1 \vee \cdots \vee B_n)$ (B_i conjunctions of prime formulas) and geometric implications $\forall_{\vec{x}}(B \to \exists_{\vec{y}}(B_1 \vee \cdots \vee B_k))$ (B, B_i conjunctions of prime formulas). Barr's theorem says that for Γ consisting of geometric implications and A a geometric formula, classical derivability of A from Γ implies intuitionistic derivability. A systematic study of such theorems has been undertaken by Orevkov [11] (cf. [7] for a good survey, [8] for very clear proofs and [8, 9, 12] for related work). We consider one Glivenko-style theorem of Orevkov, where the conclusion is $\to$-free, and the premises contain $\to$ only positively and $\vee$, $\bot$ only negatively. We give a new proof of this result (for minimal rather than intuitionistic logic), which is of interest when computational uses of classical proofs are envisaged, as in [2]. Clearly model-theoretic arguments do not help here; one needs proof transformations. But even that is not always good enough: the way proofs are represented as input data matters. In [8, 9, 11] proofs are given as derivations in a sequent calculus. However, for a computational analysis natural deduction proofs are more appropriate, since by the Curry-Howard correspondence they can directly be viewed as λ-terms. A proof of Orevkov's theorem in this setting then amounts to an analysis of possible occurrences of stability axioms, and a method to eliminate them. This is what will be done in the present paper.

In section 1 we fix our terminology for natural deduction proofs in minimal logic, and describe the standard embedding of classical logic into its $\to, \forall, \wedge$-fragment. To prepare for the proof of the main result in section 3, we recall in section 2 the relevant notions, as far as they are necessary to follow the proof. Section 4 discusses the algorithm provided by the proof, and the final section 5 gives an application[1].

1. Minimal logic

Natural deduction is a distinguished logical system, since it allows to formalize faithfully proofs done by a mathematician who wants to write out all details; this

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[1]We are grateful to Thierry Coquand for bringing this example to our attention.

Derivation	Term
$u\colon A$	u^A
$\dfrac{\begin{matrix}[u\colon A] \\ \mid M \\ B\end{matrix}}{A \to B}\,\to^+ u$	$(\lambda_{u^A} M^B)^{A\to B}$
$\dfrac{\begin{matrix}\mid M \\ A \to B\end{matrix} \qquad \begin{matrix}\mid N \\ A\end{matrix}}{B}\,\to^-$	$(M^{A\to B} N^A)^B$
$\dfrac{\begin{matrix}\mid M \\ A\end{matrix}}{\forall_x A}\,\forall^+ x$ (with var.cond.)	$(\lambda_x M^A)^{\forall_x A}$ (with var.cond.)
$\dfrac{\begin{matrix}\mid M \\ \forall_x A(x)\end{matrix} \qquad r}{A(r)}\,\forall^-$	$(M^{\forall_x A(x)} r)^{A(r)}$

TABLE 1. Derivation terms for $\to$ and $\forall$

was convincingly spelled out by Gentzen [4]. On the more technical side, natural deduction corresponds closely to the simply typed λ-calculus ("Curry-Howard correspondence"). This is particularly so if we define negation by $\neg A := A \to \bot$ with $\bot$ just a distinguished propositional symbol; the resulting system is *minimal logic*. We then can add extra axioms for $\bot$ (e.g., stability or ex-falso-quodlibet) to embed classical or intuitionistic logic. In minimal logic, for each of the connectives $\to$, $\forall$ and also $\exists$, $\vee$ and $\wedge$ we have introduction and elimination rules (I-rules and E-rules) given in tables 1 and 2. The left premise $A \to B$ in $\to^-$ is called the *major* (or *main*) premise, and the right premise A the *minor* (or *side*) premise. Similarly, in each of the elimination rules $\vee^-$, $\wedge^-$ and $\exists^-$ the left premise is called *major* (or *main*) premise, and the right premise is called the *minor* (or *side*) premise. We define the *weak* variants $\tilde{\exists}, \tilde{\vee}$ of $\exists, \vee$ by

$$\tilde{\exists}_x A := \neg\forall_x \neg A \quad \text{and} \quad A \mathbin{\tilde{\vee}} B := \neg A \to \neg B \to \bot.$$

Clearly $\vdash \exists_x A \to \tilde{\exists}_x A$ and $\vdash A \vee B \to A \mathbin{\tilde{\vee}} B$, but not conversely; this is the reason why $\tilde{\exists}, \tilde{\vee}$ are called "weak".

The *stability* axioms are of the form $\forall_{\vec{x}}(\neg\neg P\vec{x} \to P\vec{x}\,)$ with P a relation symbol distinct from $\bot$. It is easy to see that from the stability axioms we can derive $\neg\neg A \to A$ for every formula A built with $\to$, $\forall$, $\wedge$ only. Let Stab denote the set of all stability axioms. We write $\Gamma \vdash_c B$ for $\Gamma \cup \mathrm{Stab} \vdash B$, and call B *classically derivable* from Γ. Similarly, let Efq denote the set of all *ex-falso-quodlibet* axioms $\forall_{\vec{x}}(\bot \to P\vec{x}\,)$. We write $\Gamma \vdash_i B$ for $\Gamma \cup \mathrm{Efq} \vdash B$, and call B *intuitionistically*

Derivation	Term
$\dfrac{\begin{array}{c} \mid M \\ A \end{array}}{A \lor B}\,\lor_0^+ \qquad \dfrac{\begin{array}{c} \mid M \\ B \end{array}}{A \lor B}\,\lor_1^+$	$(\lor_{0,B}^+ M^A)^{A \lor B} \quad (\lor_{1,A}^+ M^B)^{A \lor B}$
$\dfrac{\begin{array}{c} \\ \mid M \\ A \lor B \end{array} \quad \begin{array}{c} [u\colon A] \\ \mid N \\ C \end{array} \quad \begin{array}{c} [v\colon B] \\ \mid K \\ C \end{array}}{C}\,\lor^- u, v$	$(M^{A \lor B}(u^A.N^C, v^B.K^C))^C$
$\dfrac{\begin{array}{c} \mid M \\ A \end{array} \quad \begin{array}{c} \mid N \\ B \end{array}}{A \land B}\,\land^+$	$\langle M^A, N^B \rangle^{A \land B}$
$\dfrac{\begin{array}{c} \\ \mid M \\ A \land B \end{array} \quad \begin{array}{c} [u\colon A] \quad [v\colon B] \\ \mid N \\ C \end{array}}{C}\,\land^- u, v$	$(M^{A \land B}(u^A, v^B.N^C))^C$
$\dfrac{r \quad \begin{array}{c} \mid M \\ A(r) \end{array}}{\exists_x A(x)}\,\exists^+$	$(\exists_{x,A}^+ r M^{A(r)})^{\exists_x A(x)}$
$\dfrac{\begin{array}{c} \\ \mid M \\ \exists_x A \end{array} \quad \begin{array}{c} [u\colon A] \\ \mid N \\ B \end{array}}{B}\,\exists^- x, u$ (var.cond.)	$(M^{\exists_x A}(x, u^A.N^B))^C$ (var.cond.)

Table 2. Derivation terms for $\lor$, $\land$ and $\exists$

derivable from Γ. Recall the *negative translation* A^g of A (due to Gödel-Gentzen), defined by

(i) $\bot^g := \bot$;
(ii) $P^g := \neg\neg P$ for prime formulas $P \neq \bot$;
(iii) $(B \lor C)^g := B^g \tilde{\lor} C^g$;
(iv) $(\exists_x B)^g := \tilde{\exists}_x B^g$;
(v) $(B \circ C)^g := B^g \circ C^g$ for $\circ = \to, \land$;
(vi) $(\forall_x B)^g := \forall_x B^g$.

It is well-known that $\Gamma \vdash_c A$ implies $\Gamma^g \vdash A^g$. The converse clearly holds for Γ, A without $\lor, \exists$.

Remark. Notice that we deal with an *extended* classical logic here: in addition to the weak ("classical") connectives $\tilde{\vee}, \tilde{\exists}$ we also have the strong ones $\vee, \exists$. Thus $\vdash P \mathrel{\tilde{\vee}} \neg P$, but $\nvdash_c P \vee \neg P$.

Traditional classical logic disregards the distinction between $\tilde{\vee}, \tilde{\exists}$ and $\vee, \exists$, which amounts to adding $A \mathrel{\tilde{\vee}} B \to A \vee B$ and $\tilde{\exists}_x A \to \exists_x A$ as axioms. Since one can easily derive (in minimal logic) $A \mathrel{\tilde{\vee}} B \leftrightarrow \neg\neg(A \vee B)$ and $\tilde{\exists}_x A \leftrightarrow \neg\neg\exists_x A$, adding these axioms is the same thing as adding stability for disjunction $\neg\neg(A \vee B) \to A \vee B$ and for existence $\neg\neg\exists_x A \to \exists_x A$.

2. Normal derivations and their structure

The Curry-Howard correspondence allows us to transform concepts well-known in λ-calculus to natural deduction proofs; this will be important later on. For the convenience of the reader we recall in this section the relevant notions, as far as they are necessary to follow the proof in section 3. For background and details we refer the reader to [13, 16, 15].

Let us first restrict to the $\to, \forall$-fragment. By a *β-redex* ("reducible expression") we mean a ($\to$ or $\forall$)-introduction immediately followed by a ($\to$ or $\forall$)-elimination. This "detour" can be simplified by a *β-conversion*:

$$\dfrac{\dfrac{\begin{matrix}[u\colon A]\\ \mid M\\ B\end{matrix}}{A \to B}\,\to^+ u \qquad \begin{matrix}\mid N\\ A\end{matrix}}{B}\,\to^- \qquad \mapsto_\beta \qquad \begin{matrix}\mid N\\ A\\ \mid M\\ B\end{matrix}$$

or written as derivation terms

$$(\lambda_u M(u^A)^B)^{A \to B} N^A \mapsto_\beta M(N^A)^B.$$

Similarly we have for the universal quantifier

$$\dfrac{\dfrac{\begin{matrix}\mid M\\ A(x)\end{matrix}}{\forall_x A(x)}\,\forall^+ x \qquad r}{A(r)}\,\forall^- \qquad \mapsto_\beta \qquad \begin{matrix}\mid M'\\ A(r)\end{matrix}$$

or written as derivation terms

$$(\lambda_x M(x)^{A(x)})^{\forall_x A(x)} r \mapsto_\beta M(r).$$

Every $\to, \forall$-proof can be *reduced* (by iterated conversions) to a *normal form*, i.e., a proof without β-redexes; this normal form is uniquely determined. To analyze the structure of normal derivations, it is useful to introduce the notion of a track in a proof tree, which makes sense for non-normal derivations as well. A *track* of a derivation M is a sequence of formula occurrences (f.o.) $A_0, \dots, A_n$ such that

(a) A_0 is a top f.o. in M (possibly discharged by an application of $\to^-$);
(b) A_i for $i < n$ is not the minor premise of an instance of $\to^-$, and A_{i+1} is directly below A_i;
(c) A_n is either the minor premise of an instance of $\to^-$, or the conclusion of M.

The *track of order* 0, or *main track*, in a derivation is the (unique) track ending in the conclusion of the whole derivation. A *track of order* $n+1$ is a track ending in the minor premise of an $\to^-$-application, with major premise belonging to a track of order n.

It is easy to see that each formula occurrence in a derivation belongs to some track. Now consider a normal derivation M. Since by normality an E-rule cannot have the conclusion of an I-rule as its major premise, the E-rules have to precede the I-rules in a track, so the following is obvious: a track may be divided into an

E-part, say $A_0, \dots, A_{i-1}$, a minimal formula A_i, and an I-part $A_{i+1}, \dots, A_n$. In the E-part all rules are E-rules; in the I-part all rules are I-rules; A_i is the conclusion of an E-rule and, if $i < n$, a premise of an I-rule. Tracks are pieces of branches of the tree with successive f.o.'s in the subformula relationship: either A_{i+1} is a subformula of A_i or vice versa. As a result, all formulas in a track $A_0, \dots, A_n$ are subformulas of A_0 or of A_n; and from this, by induction on the order of tracks, we see that every formula in M is a subformula either of an open assumption or of the conclusion. To summarize: in a normal derivation each formula is a subformula of either the end formula or else an assumption formula.

Notice that the minimal formula in a track can be an implication $A \to B$ or generalization $\forall_x A$. However, we can apply an *η-expansion* and replace the occurrence of $A \to B$ or $\forall_x A$ by

$$\dfrac{\dfrac{A \to B \qquad u\colon A}{B}\to^-}{A \to B}\to^+ u \qquad\qquad \dfrac{\dfrac{\forall_x A \qquad x}{A}\forall^-}{\forall_x A}\forall^+ x$$

Repeating this process we obtain a derivation in *long normal form*, all of whose minimal formulas are neither implications nor generalizations.

When we proceed to the full language (including $\vee, \wedge, \exists$), in addition to the $\to, \forall$-conversions we must consider the following conversions:

$\vee$-*conversion.*

$$\dfrac{\dfrac{\begin{matrix}\mid M\\ A\end{matrix}}{A \vee B}\vee_0^+ \qquad \begin{matrix}[u\colon A]\\ \mid N\\ C\end{matrix} \qquad \begin{matrix}[v\colon B]\\ \mid K\\ C\end{matrix}}{C}\vee^- u, v \qquad\mapsto\qquad \begin{matrix}\mid M\\ A\\ \mid N\\ C\end{matrix}$$

or as derivation terms $(\vee_{0,B}^+ M^A)^{A \vee B}(u^A.N(u)^C, v^B.K(v)^C) \mapsto N(M^A)^C$, and similarly for $\vee_1^+$ with K instead of N.

$\wedge$-*conversion.*

$$\dfrac{\dfrac{\begin{matrix}\mid M\\ A\end{matrix} \qquad \begin{matrix}\mid N\\ B\end{matrix}}{A \wedge B}\wedge^+ \qquad \begin{matrix}[u\colon A] \quad [v\colon B]\\ \mid K\\ C\end{matrix}}{C}\wedge^- u, v \qquad\mapsto\qquad \begin{matrix}\mid M & & \mid N\\ A & & B\\ & \mid K & \\ & C & \end{matrix}$$

or $\langle M^A, N^B\rangle^{A \wedge B}(u^A, v^B.K(u,v)^C) \mapsto K(M^A, N^B)^C$.

$\exists$-*conversion.*

$$\dfrac{\dfrac{r \qquad \begin{matrix}\mid M\\ A(r)\end{matrix}}{\exists_x A(x)}\exists^+ \qquad \begin{matrix}[u\colon A(x)]\\ \mid N\\ B\end{matrix}}{B}\exists^- x, u \qquad\mapsto\qquad \begin{matrix}\mid M\\ A(r)\\ \mid N'\\ B\end{matrix}$$

or $(\exists_{x,A}^+ r M^{A(r)})^{\exists_x A(x)}(u^{A(x)}.N(x,u)^B) \mapsto N(r, M^{A(r)})^B$.

However, there is a difficulty: an introduced formula may be used as a minor premise of an application of an elimination rule for $\vee$, $\wedge$ or $\exists$, then stay the same throughout a sequence of applications of these rules, being eliminated at the end. This also constitutes a local maximum, which we should like to eliminate; permutative conversions are designed for this situation. In a *permutative conversion* we permute an E-rule upwards over the minor premises of $\vee^-$, $\wedge^-$ or $\exists^-$. For $\vee$ we have

$$\dfrac{\dfrac{\begin{matrix}\mid M\\ A\vee B\end{matrix}\qquad \begin{matrix}\mid N\\ C\end{matrix}\qquad \begin{matrix}\mid K\\ C\end{matrix}}{C}\qquad \begin{matrix}\mid L\\ C'\end{matrix}}{D}\,\text{E-rule}\quad\mapsto$$

$$\dfrac{\begin{matrix}\mid M\\ A\vee B\end{matrix}\qquad \dfrac{\begin{matrix}\mid N\\ C\end{matrix}\qquad \begin{matrix}\mid L\\ C'\end{matrix}}{D}\,\text{E-rule}\qquad \dfrac{\begin{matrix}\mid K\\ C\end{matrix}\qquad \begin{matrix}\mid L\\ C'\end{matrix}}{D}\,\text{E-rule}}{D}$$

or with for instance $\to^-$ as E-rule

$$\begin{aligned}&(M^{A\vee B}(u^A.N^{C\to D}, v^B.K^{C\to D}))^{C\to D}L^C \mapsto\\ &(M^{A\vee B}(u^A.(N^{C\to D}L^C)^D, v^B.(K^{C\to D}L^C)^D))^D.\end{aligned}$$

For $\wedge, \exists$ there are similar permutative conversion rules.

We further need *simplification conversions*. These are somewhat trivial conversions, which remove unnecessary applications of the elimination rules for $\vee$, $\wedge$ and $\exists$. For $\vee$ we have

$$\dfrac{\begin{matrix}\\ \mid M\\ A\vee B\end{matrix}\qquad \begin{matrix}[u\colon A]\\ \mid N\\ C\end{matrix}\qquad \begin{matrix}[v\colon B]\\ \mid K\\ C\end{matrix}}{C}\,\vee^- u, v\qquad\mapsto\qquad \begin{matrix}\mid N\\ C\end{matrix}$$

if $u\colon A$ is not free in N, or $(M^{A\vee B}(u^A.N^C, v^B.K^C))^C \mapsto N^C$; similarly for the second component. For $\wedge, \exists$ there are similar simplification conversions. Again one can show that every derivation term can be reduced to a (uniquely determined) normal form where none of these conversions can be performed.

Let us now analyze the structure of normal derivations in the full language. It will be useful to introduce the notion of a segment and to modify accordingly the notion of a track in a proof tree. Both make sense for non-normal derivations as well. A *segment* (of *length* n) in a derivation M is a sequence $A_0, \dots, A_n$ of occurrences of the same formula A such that

(a) for $0 \le i < n$, A_i is a minor premise of an application of $\vee^-$, $\wedge^-$ or $\exists^-$, with conclusion A_{i+1};
(b) A_n is not a minor premise of $\vee^-$, $\wedge^-$ or $\exists^-$.
(c) A_0 is not the conclusion of $\vee^-$, $\wedge^-$ or $\exists^-$.

Notice that a formula occurrence (f.o.) which is neither a minor premise nor the conclusion of an application of $\vee^-$, $\wedge^-$ or $\exists^-$ always constitutes a segment of length 1. A segment is *maximal* or a *cut (segment)* if A_n is the major premise of an E-rule, and either $n > 0$, or $n = 0$ and $A_0 = A_n$ is the conclusion of an I-rule. We use σ, σ' for segments. σ is called a *subformula* of σ' if the formula A in σ is a subformula of B in σ'.

Notice that only $\vee^-$ is responsible for a possible branching of a segment. Segments will be linear if no $\vee^-$ is present.

The notion of a track is designed to retain the subformula property in case one passes through the major premise of an application of a $\vee^-, \wedge^-, \exists^-$-rule. In a track, when arriving at an A_i which is the major premise of an application of such a rule, we take for A_{i+1} a hypothesis discharged by this rule. More precisely, a *track* of a derivation M is a sequence of f.o.'s $A_0, \dots, A_n$ such that

(a) A_0 is a top f.o. in M not discharged by an application of a $\vee^-, \wedge^-, \exists^-$-rule;
(b) A_i for $i < n$ is not the minor premise of an instance of $\to^-$, and *either*

(i) A_i is not the major premise of an instance of a $\vee^-, \wedge^-, \exists^-$-rule and A_{i+1} is directly below A_i, *or*

(ii) A_i is the major premise of an instance of a $\vee^-, \wedge^-, \exists^-$-rule and A_{i+1} is an assumption discharged by this instance;

(c) A_n is *either*

(i) the minor premise of an instance of $\to^-$, *or*

(ii) the end formula of M, *or*

(iii) the major premise of an instance of a $\vee^-, \wedge^-, \exists^-$-rule in case there are no assumptions discharged by this instance.

It is easy to see that in a derivation each formula occurrence belongs to some track. A *track of order* 0, or *main track*, in a derivation is a track ending either in the end formula of the whole derivation or in the major premise of an application of a $\vee^-$, $\wedge^-$ or $\exists^-$-rule, provided there are no assumption variables discharged by the application. A *track of order* $n+1$ is a track ending in the minor premise of an $\to^-$-application, with major premise belonging to a track of order n. A *main branch* of a derivation is a branch π (i.e., a linearly ordered subtree) in the proof tree such that π passes only through premises of I-rules and *major premises* of E-rules, and π begins at a top node and ends in the end formula.

Since by simplification conversions we have removed every application of an $\vee^-$, $\wedge^-$ or $\exists^-$-rule that discharges no assumption variables, each track of order 0 in a normal derivation is a track ending in the end formula of the whole derivation. Note also that if we search for a main branch going upwards from the end formula, the branch to be followed is unique as long as we do not encounter an $\wedge^+$-application.

Instead of a minimal formula we now have a minimal segment on a track. Again by appropriate η-expansions we can transform each normal derivation into a long normal form, where each minimal segment has an atomic formula.

3. Removing stability axioms

Theorem 3.1. *Assume* $\Gamma \vdash_c A$, *where* Γ, A *are such that*

(i) A *is* $\to$*-free, and*

(ii) Γ *has only positive occurrences of* $\to$, *and only negative occurrences of* $\vee$ *and of* $\bot$.

Then $\Gamma \vdash A$.

Proof. We can assume that the given proof is in long normal form.

First observe that it suffices to prove the claim for a prime formula Q as goal. To see this, consider a main branch of the derivation. Its minimum segment has a prime formula, and the following introduction part has by assumption no $\to^+$-rules but $\forall^+$, $\exists^+$, $\vee^+$ and $\wedge^+$-rules only, which do not bind assumption variables.

Now use induction on the number of occurrences of stability axioms. Since the proof is in long normal form, each stability axiom must be fully eliminated, that is, it occurs in a context

(1)
$$\begin{array}{c}\dfrac{\dfrac{\mathrm{Stab}_P\colon \forall_{\vec x}(\neg\neg P\vec x \to P\vec x) \qquad \vec r}{\neg\neg P\vec r \to P\vec r} \qquad \begin{array}{c} \mid M \\ \dfrac{\bot}{\neg\neg P\vec r}\,{\to^+}u \\ \mid S \\ \neg\neg P\vec r\end{array}}{P\vec r}\,{\to^-} \\ \mid K \\ Q\end{array}$$

with a unique $\to^+ u$, because the segment tree S is linear (since it cannot contain $\vee^-$, as it would lead to a positive occurrence of $\vee$ in Γ). Call an application of a rule $\to^+ u$ *proper* if its premise has a free occurrence of u. Call an occurrence of a stability axiom *proper* if its associated $\to^+ u$ is proper.

Case 1. There is a proper occurrence of a stability axiom. Pick a topmost one. It appears in a context

$$\dfrac{\dfrac{\mathrm{Stab}_P\colon \forall_{\vec x}(\neg\neg P\vec x \to P\vec x) \qquad \vec r}{\neg\neg P\vec r \to P\vec r} \qquad \begin{array}{c} \dfrac{u\colon \neg P\vec r \qquad \begin{array}{c} \mid N \\ P\vec r \end{array}}{\bot} \\ \mid M \\ \dfrac{\bot}{\neg\neg P\vec r}\,\to^+ u \\ \mid S \\ \neg\neg P\vec r \end{array}}{\begin{array}{c} P\vec r \\ \mid K \\ Q \end{array}}\,\to^- \tag{2}$$

Since the proof is in long normal form, each leaf $u\colon \neg P\vec r$ must be the main premise of a rule $\to^-$, i.e., be in a context uN with $N\colon P\vec r$. Pick an uppermost bound occurrence of u, i.e., a subproof uN where N has no free occurrence of u.

Let $u_1, \dots, u_n$ be the assumption variables of N bound in M or S (by $\to^+$, $\exists^-$ or $\wedge^-$; $\vee^-$ would again lead to a positive occurrence of $\vee$ in Γ). Any such $\to^+$ must be in M, and the path through its conclusion $A \to B$ must end in the side premise of an $\to^-$ (since M ends with $\bot$). Its main premise contains $A \to B$ negatively. Since $A_1, \dots, A_n$ contain no negative implications, our $\to^+$ occurs as $\to^+ v$ in a context

$$\dfrac{\dfrac{\mathrm{Stab}_Q\colon \forall_{\vec x}(\neg\neg Q\vec x \to Q\vec x) \qquad \vec r}{\neg\neg Q\vec r \to Q\vec r} \qquad \begin{array}{c} \mid \\ \dfrac{\bot}{\neg\neg Q\vec r}\,\to^+ v \end{array}}{\begin{array}{c} Q\vec r \\ \mid \end{array}}\,\to^-$$

and hence is an improper application, since we picked Stab_P as a topmost proper occurrence of a stability axiom. Therefore $u_1, \dots, u_n$ must all be bound by $\exists^-$ or $\wedge^-$. Consider the main parts of these $\exists^-$, $\wedge^-$. We push them all up to the end of N, i.e., leave the main parts as they are, but use them with a new side part, each with side formula $P\vec r$. This new side formula does not affect the validity of the variable condition at any such $\exists^-$, since in each of them $u\colon \neg P\vec r$ was an open assumption. Thus all $u_1, \dots, u_n$ get bound, and u has disappeared. Let $\hat N$ be this extension of N. The result is

$$\begin{array}{c} \mid \hat N \\ P\vec r \\ \mid K \\ Q \end{array}$$

We have removed one occurrence of a stability axiom and can apply the induction hypothesis.

Case 2. There are only improper occurrences of stability axioms. Assume there is one. It must appear in a context (1). Let A be the formula of a topmost node in a path through the end formula $\bot$ of M; it has $\bot$ as a strictly positive part. Since we are in case 2, this topmost node cannot be bound by $\to^+$. Hence $\bot$ must be a strictly positive part of a formula in Γ, which contradicts our assumptions. $\square$

None of the assumptions on A, Γ in Theorem 3.1 can be omitted:

(i) "A is $\to$-free". The Peirce formula provides a countereranple: we have $\vdash_c ((P \to Q) \to P) \to P$, but $\nvdash_i ((P \to Q) \to P) \to P$.

(ii) "Γ has only positive occurrences of $\to$". Again we can use the Peirce formula: $(P \to Q) \to P \vdash_c P$, but $(P \to Q) \to P \nvdash_i P$.

(iii) "Γ has only negative occurrences of $\vee$". This example is due to Nadathur [8].

$$\forall_x Px \to Q, \forall_x (Px \vee Q) \vdash_c Q, \quad \text{but} \quad \forall_x Px \to Q, \forall_x (Px \vee Q) \nvdash_i Q.$$

To see where the argument above breaks down, consider the derivation from Stab_Q and the consequence Efq_P of Stab_P:

$$\dfrac{\mathrm{Stab}_Q\colon \neg\neg Q \to Q \qquad \dfrac{\begin{array}{c} \mid M \\ \bot \end{array}}{\neg\neg Q}\to^+ u}{Q}$$

where M is

$$\dfrac{u{:}\neg Q \qquad \dfrac{\forall_x Px \to Q \qquad \dfrac{\dfrac{\dfrac{\forall_x(Px \vee Q) \quad x}{Px \vee Q} \qquad v{:}Px \qquad \dfrac{\dfrac{\mathrm{Efq}_P\colon \forall_x(\bot \to Px) \quad x}{\bot \to Px} \qquad \dfrac{u{:}\neg Q \quad w{:}Q}{\bot}}{Px}}{Px}\vee^- v, w}{\forall_x Px}}{Q}}{\bot}$$

Here we cannot push up $\vee^-$ changing its side formula to Q, since it has *two* side premises.

(iv) "Γ has only negative occurrences of $\bot$". $\bot \vdash_c P$, but $\bot \nvdash P$.

A special case of Theorem 3.1 was proved in [14], with a similar method. Define a *generalized definite Horn formula* to be $\forall_{\vec{x}}(C_1 \to \cdots \to C_n \to B)$ with C_i of the form $\forall_{\vec{y}_i} B_i$ and B_i, B prime formulas with B distinct from $\bot$. The proof in [14] shows that for Γ consisting of generalized definite Horn formulas and A a prime formula, $\Gamma \vdash_c A$ implies $\Gamma \vdash A$. Related extensions of logic programming have been studied in detail by Dale Miller [6].

4. An algorithm to remove stability axioms

We describe the algorithm contained in the proof of Theorem 3.1. Recall that the input has to be a derivation tree in *long normal form.* The algorithm essentially consists in two depth-first passes through the derivation tree. The outer one identifies proper applications of stability axioms, which must be of the form (2). Let u be the assumption variable bound by the associated $\to^+ u$. Then a second depth-first pass (depending on u) through the side premise (ending in $\neg\neg P\vec{r}$) of its $\to^-$ is done, resulting in a derivation tree for $P\vec{r}$ with no free occurrence of u. Thus we get rid of all proper applications of stability axioms, and as argued in case 2 of the proof no improper occurrences of stability axioms remain. Thus we have one primitive recursion inside another one, and hence the algorithm is quadratic relative to the length of the proof tree.

We now give a more detailed description. Clearly we can assume that all bound assumption variables in the given derivation are distinct. In a first step we mark each subderivation with its free assumption variables, and also one bit indicating whether or not we have a proper application of a stability axiom. Given this data we recursively define a reduction function eliminating all proper applications of stability. If the derivation does not end with a proper application of stability, take the same rule and recursively apply the reduction function to the subderivations. Hence we can assume that our derivation ends with a proper application of a stability axiom

$$\cfrac{\cfrac{\mathrm{Stab}_P\colon \forall_{\vec{x}}(\neg\neg P\vec{x} \to P\vec{x}\,) \qquad \vec{r}}{\neg\neg P\vec{r} \to P\vec{r}} \qquad \begin{array}{c} \mid L \\ \neg\neg P\vec{r} \end{array}}{P\vec{r}} \to^-$$

Hence L must be of the form

$$\begin{array}{c} \mid \\ \cfrac{\bot}{\neg\neg P\vec{r}} \to^+ u \\ \mid S \\ \neg\neg P\vec{r} \end{array}$$

with a segment S, and we can read off u from L. Mark each subderivation of L according to whether it contains u (a) neither free nor bound, (b) free or (c) bound. We define a second reduction function $\mathrm{Red}_u(L)$ depending on u by recursion on L (using notation from (2)).

(i) $\mathrm{Red}_u(L) := L$ if u is neither free nor bound in L.

(ii)

$$\cfrac{u\colon \neg P\vec{r} \qquad \begin{array}{c} \mid L \\ P\vec{r} \end{array}}{\bot} \to^- \qquad \mapsto_{\mathrm{Red}_u} \qquad \begin{array}{c} \mid \mathrm{Red}_u(L) \\ P\vec{r} \end{array}$$

if u is neither free nor bound in L.

(iii)

$$\cfrac{\begin{array}{c} \mid \\ \exists_x A \end{array} \qquad \begin{array}{c} \mid L \\ \bot \end{array}}{\bot} \exists^- \qquad \mapsto_{\mathrm{Red}_u} \qquad \cfrac{\begin{array}{c} \mid \\ \exists_x A \end{array} \qquad \begin{array}{c} \mid \mathrm{Red}_u(L) \\ P\vec{r} \end{array}}{P\vec{r}} \exists^-$$

if u is free in L. Similarly for $\wedge^-$.

(iv)

$$\cfrac{\begin{array}{c} \mid L \\ \bot \end{array}}{\neg\neg P\vec{r}} \to^+ u \qquad \mapsto_{\mathrm{Red}_u} \qquad \begin{array}{c} \mid \mathrm{Red}_u(L) \\ P\vec{r} \end{array}$$

if u is free in L.

(v)

$$\cfrac{\begin{array}{c} \mid \\ \exists_x A \end{array} \qquad \begin{array}{c} \mid L \\ \neg\neg P\vec{r} \end{array}}{\neg\neg P\vec{r}} \exists^- \qquad \mapsto_{\mathrm{Red}_u} \qquad \cfrac{\begin{array}{c} \mid \\ \exists_x A \end{array} \qquad \begin{array}{c} \mid \mathrm{Red}_u(L) \\ P\vec{r} \end{array}}{P\vec{r}} \exists^-$$

if u is bound in L. Similarly for $\wedge^-$.

(vi)

$$\cfrac{\cfrac{\mathrm{Stab}_P\colon \forall_{\vec{x}}(\neg\neg P\vec{x} \to P\vec{x}\,) \qquad \vec{r}}{\neg\neg P\vec{r} \to P\vec{r}} \qquad \begin{array}{c} \mid L \\ \neg\neg P\vec{r} \end{array}}{P\vec{r}} \to^- \qquad \mapsto_{\mathrm{Red}_u} \qquad \begin{array}{c} \mid \mathrm{Red}_u(L) \\ P\vec{r} \end{array}$$

if u is bound in L.

5. An application in algebra

For applications of Theorem 3.1 we need a formalization of the proposition at hand in first order logic, in the form $\Gamma \vdash_c A$ with Γ, A satisfying the restrictions of

Theorem 3.1. A good candidate for inclusion in Γ is the theory $\mathrm{Ax}_{\mathrm{ring}}$ of commutative rings, in the language given by function symbols $+, \times, -$ and constants $0, 1$ and with axioms (writing xy for $x \times y$)

$$x + (y + z) = (x + y) + z, \quad x + y = y + x, \quad x + 0 = x, \quad x + (-x) = 0,$$
$$x(yz) = (xy)z, \quad xy = yx, \quad x1 = x, \quad x(y + z) = xy + xz.$$

Notice that the theory of integral rings with the additional axiom

$$xy = 0 \to x = 0 \lor y = 0$$

does not qualify, because of the positive occurrence of $\lor$.

In spite of the fact that many concepts and proof methods common in algebra (Noetherian rings, Zorn's lemma etc.) go beyond first order logic, recent work of Coquand, Lombardi and others has revealed that in many cases one can find substitutes allowing a formalization in first order logic. We have to refer to the literature (in particular [3]) for background and definitions, and restrict ourselves to a discussion of one example in [3], a non-Noetherian version of Swan's theorem. It is written in a purely first order way as an implication

$$\mathrm{Ax}_{\mathrm{ring}} \to \mathrm{Hdim}R < n \to \Delta_n(F) = 1 \to \exists_{X,Y}(1 = XFY) \tag{3}$$

where X is a row vector, Y a column vector and F a matrix of fixed size. We refer to [3] for how $\mathrm{Hdim}R < n$ (the Heitmann dimension of a ring R is $< n$) can be written as a first-order formula; for $n = 1$ it means

$$\forall_x \exists_a \forall_y \exists_b (1 = b(1 - yx(1 - ax))).$$

The logical complexity increases with n, but $\mathrm{Hdim}R < n$ (for a given n) can still be written as a prenex formula with alternating quantifiers and an equational kernel, and hence is $\to$-free. $\Delta_n(F)$ is the ideal generated from all minors of F of order n. Hence $\Delta_n(F) = 1$ is an existential formula with an equational kernel. The same is true for the conclusion $\exists_{X,Y}(1 = XFY)$. Therefore Theorem 3.1 applies, telling us how to remove stability axioms (for atomic formulas, i.e., equations). Thus we obtain a proof of (3) in minimal logic.

References

[1] M. Barr. Toposes without points. *J. Pure and Applied Algebra*, 5:265–280, 1974.

[2] U. Berger, W. Buchholz, and H. Schwichtenberg. Refined program extraction from classical proofs. *Annals of Pure and Applied Logic*, 114:3–25, 2002.

[3] T. Coquand and H. Lombardi. A logical approach to abstract algebra. *Math. Struct. in Comp. Science*, 16:885–900, 2006.

[4] G. Gentzen. Untersuchungen über das logische Schließen I, II. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1935.

[5] V. Glivenko. Sur quelques points de la logique de M. Brouwer. *Bull. Soc. Math. Belg.*, 15:183–188, 1929.

[6] D. Miller. A logic programming language with lambda–abstraction, function variables and simple unification. *Journal of Logic and Computation*, 2(4):497–536, 1991.

[7] G. Mints. Proof theory in the USSR 1925–1969. *The Journal of Symbolic Logic*, 56(2):385–424, 1991.

[8] G. Nadathur. Correspondence between classical, intuitionistic and uniform provability. *Theoretical Computer Science*, 232(1–2):273–298, 2000.

[9] S. Negri. Contraction-free sequent calculi for geometric theories with an application to Barr's theorem. *Archive for Mathematical Logic*, 42:389–401, 2003.

[10] V. P. Orevkov, editor. *Logical and logico-mathematical calculi. I.* English translation, The calculi of symbolic logic. I, Proceedings of the Steklov Institute of Mathematics, vol. 98 (1971). Trudy Matematicheskogo Instituta imeni V.A. Steklova, 1968.

[11] V. P. Orevkov. On Glivenko sequent classes. In *Logical and logico-mathematical calculi* [10], pages 131–154 (Russian), 147–173 (English).

[12] E. Palmgren. An intuitionistic axiomatization of real closed fields. *Math Log. Quart.*, 48(2):297–299, 2002.

[13] D. Prawitz. Ideas and results in proof theory. In J. Fenstad, editor, *Proceedings of the Second Scandinavian Logic Symposium*, pages 235–307. North-Holland, Amsterdam, 1971.

[14] H. Schwichtenberg. Minimal from classical proofs. In E. Börger, G. Jäger, H. Kleine-Büning, and M. Richter, editors, *Computer Science Logic*, volume 626 of *LNCS*, pages 326–328. Springer Verlag, Berlin, Heidelberg, New York, 1992.

[15] H. Schwichtenberg and S. S. Wainer. *Proofs and Computations*. Perspectives in Logic. Association for Symbolic Logic and Cambridge University Press, 2012.

[16] A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics. An Introduction*, volume 121, 123 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 1988.

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