Note that the examples on page 37 are no longer critical: The first one yields $z : \alpha \to \forall\gamma\beta \vdash \lambda x^\alpha.zx\gamma : \alpha \to \beta$, the second $z : \alpha \to \beta \vdash \lambda x^\alpha \Lambda\gamma.zx : \alpha \to \forall\gamma\beta$. In both cases, we cannot apply an outer η-reduction.

Before studying confluence and strong normalization of the typable terms, we consider several examples showing the expressivity of system F.

Examples 10 *1. Set* $0 := \forall\alpha\alpha$. *Then* $\Gamma \vdash r : 0$ *implies* $\Gamma \vdash r\rho : \rho$.

2. *Set* $1 := \forall\alpha.\alpha \to \alpha$ *and* $\mathsf{IN1} := \Lambda\alpha\lambda x^\alpha x$. *Then* $\vdash \mathsf{IN1} : 1$.

3. *Set* $\rho \times \sigma := \forall\alpha.(\rho \to \sigma \to \alpha) \to \alpha$ *for some* $\alpha \notin FV(\rho) \cup FV(\sigma)$. *Set* $\langle r, s\rangle_{\rho,\sigma} := \Lambda\alpha\lambda z^{\rho\to\sigma\to\alpha}.zrs$ *for some* $z \notin FV(r) \cup FV(s)$ *and assume that* $\alpha \notin FTV(r) \cup FTV(s)$. *Hence, if* $\Gamma \vdash r : \rho$ *and* $\Gamma \vdash s : \sigma$ *then* $\Gamma \vdash \langle r, s\rangle_{\rho,\sigma} : \rho \times \sigma$. *Set* $r\mathsf{L}_{\rho,\sigma} := r\rho(\lambda x^\rho\lambda y^\sigma.x)$ *and* $r\mathsf{R}_{\rho,\sigma} := r\sigma(\lambda x^\rho\lambda y^\sigma.y)$. *Hence, if* $\Gamma \vdash r : \rho \times \sigma$ *then* $\Gamma \vdash r\mathsf{L}_{\rho,\sigma} : \rho$ *and* $\Gamma \vdash r\mathsf{R}_{\rho,\sigma} : \sigma$. *Moreover,* $\langle r, s\rangle_{\rho,\sigma}\mathsf{L}_{\rho,\sigma} \to^*_{\beta\eta} r$ *and* $\langle r, s\rangle_{\rho,\sigma}\mathsf{R}_{\rho,\sigma} \to^*_{\beta\eta} s$.

4. *Set* $\rho + \sigma := \forall\alpha.(\rho \to \alpha) \to (\sigma \to \alpha) \to \alpha$ *for some* $\alpha \notin FV(\rho) \cup FV(\sigma)$. *Set* $\mathsf{INL}_{\rho,\sigma}r := \Lambda\alpha\lambda x^{\rho\to\alpha}\lambda y^{\sigma\to\alpha}.xr$ *and* $\mathsf{INR}_{\rho,\sigma}r := \Lambda\alpha\lambda x^{\rho\to\alpha}\lambda y^{\sigma\to\alpha}.yr$ *for some* $x, y \notin FV(r)$ *(we assume that* $\alpha \notin FTV(r)$*). Hence, if* $\Gamma \vdash r : \rho$ *then* $\Gamma \vdash \mathsf{INL}_{\rho,\sigma}r : \rho + \sigma$, *and if* $\Gamma \vdash r : \sigma$ *then* $\Gamma \vdash \mathsf{INR}_{\rho,\sigma}r : \rho + \sigma$. *Moreover, if* $\Gamma \vdash r : \rho + \sigma$, $\Gamma \vdash s : \rho \to \tau$ *and* $\Gamma \vdash t : \sigma \to \tau$ *then* $\Gamma \vdash r\tau st : \tau$ *which gives a construct for case distinction as follows:* $\mathsf{INL}_{\rho,\sigma}r\tau st \to^*_{\beta\eta} sr$ *and* $\mathsf{INR}_{\rho,\sigma}r\tau st \to^*_{\beta\eta} tr$.

5. *Set* $\exists\alpha\rho := \forall\beta.(\forall\alpha.\rho \to \beta) \to \beta$ *for some* $\beta \notin \{\alpha\} \cup FV(\rho)$. *Set* $\mathsf{C}_{\exists\alpha\rho,\tau}r := \Lambda\beta\lambda x^{\forall\alpha.\rho\to\beta}.x\tau r$ *for some* $x \notin FV(r)$ *and* $\beta \notin FTV(r) \cup FV(\tau)$. *Hence, if* $\Gamma \vdash r : \rho[\alpha := \tau]$ *then* $\Gamma \vdash \mathsf{C}_{\exists\alpha\rho,\tau}r : \exists\alpha\rho$. *Also, if* $\Gamma \vdash r : \exists\alpha\rho$ *and* $\Gamma \vdash s : \forall\alpha.\rho \to \sigma$ *with* $\alpha \notin FV(\sigma)$, *then* $\Gamma \vdash r\sigma s : \sigma$. *Moreover,* $\mathsf{C}_{\exists\alpha\rho,\tau}r\sigma s \to^*_{\beta\eta} s\tau r$.

6. *Set* $\mathsf{nat} := \forall\alpha.(\alpha \to \alpha) \to \alpha \to \alpha$. *Set* $0 := \Lambda\alpha\lambda x^{\alpha\to\alpha}\lambda y^\alpha y$ *(this should not be confused with the type* $0 = \forall\alpha\alpha$*). Then* $\vdash 0 : \mathsf{nat}$. *Set* $\mathsf{S}r := \Lambda\alpha\lambda x^{\alpha\to\alpha}\lambda y^\alpha.x(r\alpha xy)$ *for some* $x, y \notin FV(r)$ *and* $\alpha \notin FTV(r)$. *Hence, if* $\Gamma \vdash r : \mathsf{nat}$ *then* $\Gamma \vdash \mathsf{S}r : \mathsf{nat}$. *This gives back iteration on naturals as follows: If* $\Gamma \vdash r : \mathsf{nat}$, $\Gamma \vdash s : \rho \to \rho$ *and* $\Gamma \vdash t : \rho$ *then* $\Gamma \vdash r\rho st : \rho$. *Moreover,* $0\rho st \to^*_{\beta\eta} t$ *and* $(\mathsf{S}r)\rho st \to^*_{\beta\eta} s(r\rho st)$, *hence* t *is the initial term of the iteration, and* s *is the step term.*

Lemma 42 (Typed local confluence) *If* $\Gamma \vdash r : \rho$, $r \to_{\beta\eta} r'$ *and* $r \to_{\beta\eta} r''$ *then there is a term* t *such that* $r' \to_{\beta\eta} t$ *and* $r'' \to_{\beta\eta} t$.

Proof Induction on r, case distinction according to the last rule of $\to_{\beta\eta}$ used to establish $r \to_{\beta\eta} r'$ and $r \to_{\beta\eta} r''$. Hence, we have to distinguish 81 cases.

The 9 cases in which the same rule is applied in both reductions either hold trivially (in the initial cases) or are immediate by the induction hypothesis. The other 72 cases come in 36 pairs of symmetric situations. We only consider the pairs where the first rule comes first in the list of rules. Note that the following 30 cases are syntactically impossible:

- β with η, β_F, η_F, ξ, ξ_F and t
- η with β_F, η_F, ξ_F, r, l and t
- β_F with η_F, ξ, ξ_F, r and l
- η_F with ξ, r, l and t
- ξ with ξ_F, r, l and t
- ξ_F with r, l and t
- r with t
- l with t

Hence, only 6 cases have to be considered:

β/r. We have with $\lambda x^\rho r \to_{\beta\eta} t$.

$$\begin{tikzcd} & (\lambda x^\rho r)s \arrow[dl, "\beta\eta"'] \arrow[dr, "\beta\eta"] & \\ r[x := s] & & ts \end{tikzcd}$$

If $t = \lambda x^\rho r'$ with $r \to_{\beta\eta} r'$ then by substitutivity $r[x := s] \to_{\beta\eta} r'[x := s]$, hence we get

$$\begin{tikzcd} r[x := s] \arrow[ddr, "\beta\eta"'] & & ts \arrow[d, equal] \\ & & (\lambda x^\rho r')s \arrow[dl, "\beta\eta"] \\ & r'[x := s] & \end{tikzcd}.$$

Otherwise, $r = tx$ with $x \notin FV(t)$. Then $r[x := s] = t[x := s]s = ts$.

β/l. Use compatibility as in the case l/β in the proof of Lemma 5.

η/ξ. We have with $rx \to_{\beta\eta} t$.

$$\begin{tikzcd} & \lambda x^\rho.rx \arrow[dl, "\beta\eta"'] \arrow[dr, "\beta\eta"] & \\ r & & \lambda x^\rho t \end{tikzcd}$$

If $t = r'x$ with $r \to_{\beta\eta} r'$, then also $x \notin FV(r')$, hence $\lambda x^\rho t \to_{\beta\eta} r'$, yielding

$$\begin{tikzcd} r \arrow[dr, "\beta\eta"'] & & \lambda x^\rho t \arrow[dl, "\beta\eta"] \\ & r' & \end{tikzcd}.$$

Otherwise, $r = \lambda y^\sigma s$ and $t = s[y := x]$. Since $\lambda x^\rho.rx$ is typable[12], there are Γ and τ such that $\Gamma \vdash \lambda x^\rho.(\lambda y^\sigma s)x : \rho \to \tau$. This comes from $\Gamma, x : \rho \vdash (\lambda y^\sigma s)x : \tau$, hence $\Gamma, x : \rho \vdash \lambda y^\sigma s : \rho \to \tau$. We conclude $\rho = \sigma$. Finally, since $x \notin FV(r)$, $\lambda x^\rho t = \lambda x^\sigma.s[y := x] = \lambda y^\sigma s = r$.

[12]This is the only place where we need the typability assumption.

β_F/t. We have

$$\begin{tikzcd}[column sep=small] & (\Lambda\alpha r)\sigma \arrow[dl, "\beta\eta"'] \arrow[dr, "\beta\eta"] & \\ r[\alpha := \sigma] & & t\sigma \end{tikzcd}$$

with $\Lambda\alpha r \to_{\beta\eta} t$.

If $t = \Lambda\alpha r'$ with $r \to_{\beta\eta} r'$ then by Lemma 37, $r[\alpha := \sigma] \to_{\beta\eta} r'[\alpha := \sigma]$, hence we get

$$\begin{tikzcd}[column sep=small] r[\alpha := \sigma] \arrow[ddr, "\beta\eta"'] & & t\sigma \arrow[d, equal] \\ & & (\Lambda\alpha r')s \arrow[dl, "\beta\eta"] \\ & r'[\alpha := \sigma] & \end{tikzcd}.$$

Otherwise, $r = t\alpha$ with $\alpha \notin \mathrm{FTV}(t)$, hence $r[\alpha := \sigma] = t[\alpha := \sigma]\sigma = t\sigma$.

η_F/ξ_F. We have

$$\begin{tikzcd}[column sep=small] & \Lambda\alpha.r\alpha \arrow[dl, "\beta\eta"'] \arrow[dr, "\beta\eta"] & \\ r & & \Lambda\alpha t \end{tikzcd}$$

with $r\alpha \to_{\beta\eta} t$.

If $t = r'\alpha$ with $r \to_{\beta\eta} r'$ then also $\alpha \notin \mathrm{FTV}(r')$, hence $\Lambda\alpha t \to_{\beta\eta} r'$, yielding

$$\begin{tikzcd}[column sep=small] r \arrow[dr, "\beta\eta"'] & & \Lambda\alpha t \arrow[dl, "\beta\eta"] \\ & r' & \end{tikzcd}.$$

Otherwise, $r = \Lambda\beta s$ and $t = s[\beta := \alpha]$. If $\alpha = \beta$, then we are done. Otherwise, $\alpha \notin \mathrm{FTV}(s)$ and hence also $\Lambda\alpha t = \Lambda\beta s$ by renaming of the bound type variable.

r/l. See the same case in the proof of Lemma 5. $\square$

In order to prove that even typed confluence holds, we first establish strong normalization: First we redefine the sets SN and SAT, and then show essentially the same results for SN and SAT as for the system of universal types.[13]

Definition 33 *Define the set* SN *inductively by:*

- *If the terms among* $\vec{R}$ *are in* SN *then* $x\vec{R} \in \mathsf{SN}$.
- *If* $r \in \mathsf{SN}$ *then* $\lambda x^\rho r \in \mathsf{SN}$.
- *If* $r \in \mathsf{SN}$ *then* $\Lambda\alpha r \in \mathsf{SN}$.
- *If* $r[x := s]\vec{S} \in \mathsf{SN}$ *and* $s \in \mathsf{SN}$ *then* $(\lambda x^\rho r)s\vec{S} \in \mathsf{SN}$.
- *If* $r[\alpha := \sigma]\vec{S} \in \mathsf{SN}$ *then* $(\Lambda\alpha r)\sigma\vec{S} \in \mathsf{SN}$.

[13] One could also derive strong normalization from that of the system of universal types. We prefer the direct proof, since it will be extended to the system of fixed-point types.

Definition 34 *A set* $\mathcal{M}$ *of terms of system* F *is saturated if the following holds:*

1. *If* $r \in \mathcal{M}$ *then* $r \in \mathsf{SN}$.
2. *If the terms among* $\vec{R}$ *are in* SN *then* $x\vec{R} \in \mathcal{M}$.
3. *If* $r[x := s]\vec{S} \in \mathcal{M}$ *and* $s \in \mathsf{SN}$ *then* $(\lambda x r)s\vec{S} \in \mathcal{M}$.
4. *If* $r[\alpha := \sigma]\vec{S} \in \mathcal{M}$ *then* $(\Lambda\alpha r)\sigma\vec{S} \in \mathcal{M}$.

Let again SAT be the set of saturated sets. Again, $\mathsf{SN} \in \mathsf{SAT}$.

Lemma 43 $\mathsf{SN} \subseteq \mathsf{sn} := \mathsf{acc}_{\to_{\beta\eta}}$.

Proof We have to show that sn has all the defining properties of SN, i. e., we have to show that

- If the terms among $\vec{R}$ are in sn then $x\vec{R} \in \mathsf{sn}$.
- If $r \in \mathsf{sn}$ then $\lambda x^\rho r \in \mathsf{sn}$.
- If $r \in \mathsf{sn}$ then $\Lambda\alpha r \in \mathsf{sn}$.
- If $r[x := s]\vec{S} \in \mathsf{sn}$ and $s \in \mathsf{sn}$ then $(\lambda x^\rho r)s\vec{S} \in \mathsf{sn}$.
- If $r[\alpha := \sigma]\vec{S} \in \mathsf{sn}$ then $(\Lambda\alpha r)\sigma\vec{S} \in \mathsf{sn}$.

The case with the variable in the head is obvious since every reduction in $x\vec{R}$ takes place in one of the $\vec{R}$.

The abstraction case is more complicated: Do induction on $r \in \mathsf{sn}$. Assume $\lambda x^\rho r \to_{\beta\eta} t$. Either $t = \lambda x^\rho r'$ with $r \to_{\beta\eta} r'$, and $t \in \mathsf{sn}$ by induction hypothesis, or $r = tx$ and $t \in \mathsf{sn}$ since it is a subterm of $r \in \mathsf{sn}$. A similar argument is needed for the Λ-abstraction.

The case of $(\lambda x^\rho r)s\vec{S}$ is as for untyped lambda calculus (see the proof of Lemma 18): By main induction on $s \in \mathsf{sn}$ and side induction on $r[x := s]\vec{S} \in \mathsf{sn}$ we prove that $(\lambda x^\rho r)s\vec{S} \in \mathsf{sn}$. Therefore, we have to show for every t with $(\lambda x^\rho r)s\vec{S} \to_{\beta\eta} t$ that $t \in \mathsf{sn}$. The only new case compared to the treatment of untyped lambda calculus is $r = r'x$ with $x \notin \mathrm{FV}(r')$ and an η-reduction applied to $\lambda x^\rho.r'x$, leading to $r's\vec{S}$. However, this is not really a new case, since β-reduction of $(\lambda x^\rho.r'x)s$ also yields $(\lambda x^\rho r)s\vec{S} \to_{\beta\eta} r'[x := s]s\vec{S} = r's\vec{S}$.

Finally consider $(\Lambda\alpha r)\sigma\vec{S}$. Show that this term is in sn by induction on $r[\alpha := \sigma]\vec{S} \in \mathsf{sn}$. Assume that $(\Lambda\alpha r)\sigma\vec{S} \to_{\beta\eta} t$. Show that $t \in \mathsf{sn}$. Either $t = (\Lambda\alpha r)\sigma\vec{S}$ with $r \to_{\beta\eta} r'$, hence $r[\alpha := \sigma]\vec{S} \to_{\beta\eta} r'[\alpha := \sigma]\vec{S}$, and we are done by the induction hypothesis. Or $t = (\Lambda\alpha r)\sigma\vec{S}'$ with reduction of one of the terms in $\vec{S}$. Then $r[\alpha := \sigma]\vec{S} \to_{\beta\eta} r[\alpha := \sigma]\vec{S}'$, and again the induction hypothesis applies. Or $t = r[\alpha := \sigma]\vec{S} \in \mathsf{sn}$. Or, finally, $r = r'\alpha$ with $\alpha \notin \mathrm{FTV}(r')$ and $t = r'\sigma\vec{S}$. But $t = (r'x)[\alpha := \sigma]\vec{S} \in \mathsf{sn}$ by assumption. $\square$

Note that $\mathsf{SN} = \mathsf{sn}$ does not hold (see Lemma 39).

Again, it is always possible to produce a saturated set from an arbitrary set M of terms by the saturated closure $\mathsf{cl}(M)$, defined by induction as follows:

- If $r \in M \cap \mathsf{SN}$ then $r \in \mathsf{cl}(M)$.
- If the terms among $\vec{R}$ are in SN then $x\vec{R} \in \mathsf{cl}(M)$.
- If $r[x := s]\vec{S} \in \mathsf{cl}(M)$ and $s \in \mathsf{SN}$ then $(\lambda x^\rho r)s\vec{S} \in \mathsf{cl}(M)$.
- If $r[\alpha := \sigma]\vec{S} \in \mathsf{cl}(M)$ then $(\Lambda\alpha r)\sigma\vec{S} \in \mathsf{cl}(M)$.

Since, again, $\mathsf{cl}(M) \subseteq \mathsf{SN}$ (proved by induction on the definition), it is the least saturated set containing $M \cap \mathsf{SN}$.

Given a saturated set $\mathcal{M}$ and a saturated set $\mathcal{N}$, we construct a saturated set $\mathcal{M} \to \mathcal{N}$: Define

$$\mathsf{S}_x(\mathcal{M},\mathcal{N}) := \{r \mid \forall s \in \mathcal{M}\; r[x := s] \in \mathcal{N}\},$$

$$\mathsf{I}(\mathcal{M},\mathcal{N}) := \{\lambda x^\rho r \mid x \in \mathcal{V},\ \rho \in \mathcal{T}_u \text{ and } r \in \mathsf{S}_x(\mathcal{M},\mathcal{N})\} \quad \text{and}$$

$$\mathsf{E}(\mathcal{M},\mathcal{N}) := \{r \mid \forall s \in \mathcal{M}, rs \in \mathcal{N}\}.$$

We get the introduction-based definition $\mathcal{M} \to_{\mathrm{I}} \mathcal{N}$ and the elimination-based definition $\mathcal{M} \to_{\mathrm{E}} \mathcal{N}$ of saturated sets:

$$\mathcal{M} \to_{\mathrm{I}} \mathcal{N} := \mathsf{cl}(\mathsf{I}(\mathcal{M},\mathcal{N})) \text{ and } \mathcal{M} \to_{\mathrm{E}} \mathcal{N} := \mathsf{cl}(\mathsf{E}(\mathcal{M},\mathcal{N})).$$

We get the same properties as before (compare Lemma 31).

Lemma 44 $\mathsf{I}(\mathcal{M},\mathcal{N}) \subseteq \mathsf{SN}$, $\mathsf{E}(\mathcal{M},\mathcal{N}) \cap \mathsf{SN} \in \mathsf{SAT}$, *and* $\mathsf{I}(\mathcal{M},\mathcal{N}) \subseteq \mathsf{E}(\mathcal{M},\mathcal{N})$.

Proof

(1) See the proof of Lemma 31.

(2) Check the conditions of saturatedness for $\mathsf{E}(\mathcal{M},\mathcal{N}) \cap \mathsf{SN}$:

1. Trivial.
2. Let the terms among $\vec{R}$ be in SN and $s \in \mathcal{M}$. Since $s \in \mathsf{SN}$ and $\mathcal{N} \in \mathsf{SAT}$, $x\vec{R}s \in \mathcal{N}$.

3./4. Simply append s and use saturatedness of $\mathcal{N}$.

(3) See the proof of Lemma 31. □

From the lemma, we get $\mathsf{I}(\mathcal{M},\mathcal{N}) \subseteq \mathcal{M} \to_{\mathrm{I}} \mathcal{N}$, $\mathcal{M} \to_{\mathrm{E}} \mathcal{N} = \mathsf{E}(\mathcal{M},\mathcal{N}) \cap \mathsf{SN}$ and, due to monotonicity of cl, $\mathcal{M} \to_{\mathrm{I}} \mathcal{N} \subseteq \mathcal{M} \to_{\mathrm{E}} \mathcal{N}$.

Define $\mathcal{M} \to \mathcal{N} := \mathcal{M} \to_{\mathrm{X}} \mathcal{N}$ with $\mathrm{X} \in \{\mathrm{I}, \mathrm{E}\}$. As before, we never use any property depending on this choice but only the following three properties which are valid for both choices and follow immediately from the preceding remarks:

(SAT) $\mathcal{M} \to \mathcal{N} \in \mathsf{SAT}$.

$(\to_{\mathrm{I}})$ If $r \in \mathsf{S}_x(\mathcal{M},\mathcal{N})$ then $\lambda x^\rho r \in \mathcal{M} \to \mathcal{N}$.

$(\to_{\mathrm{E}})$ $r \in \mathcal{M} \to \mathcal{N} \wedge s \in \mathcal{M} \Rightarrow rs \in \mathcal{N}$.

Since we now have an explicit $\forall$-introduction and $\forall$-elimination in the term system, we also have to provide an explicit construction of universal quantification on saturated sets. Given a function Φ from SAT to SAT, we define

$$\mathsf{S}_\alpha(\Phi) := \{r \mid \forall \sigma \in \mathcal{T}_u \forall \mathcal{M} \in \mathsf{SAT}\; r[\alpha := \sigma] \in \Phi(\mathcal{M})\},$$

$$\mathsf{I}(\Phi) := \{\Lambda\alpha r \mid \alpha \in \mathcal{V}_\mathrm{T} \text{ and } r \in \mathsf{S}_\alpha(\Phi)\} \quad \text{and}$$

$$\mathsf{E}(\Phi) := \{r \mid \forall \sigma \in \mathcal{T}_u \forall \mathcal{M} \in \mathsf{SAT}\; r\sigma \in \Phi(\mathcal{M})\}.$$

We get the introduction-based definition $\forall_{\mathrm{I}}\Phi$ and the elimination-based definition $\forall_{\mathrm{E}}\Phi$ of saturated sets:

$$\forall_{\mathrm{I}}\Phi := \mathsf{cl}(\mathsf{I}(\Phi)) \text{ and } \forall_{\mathrm{E}}\Phi := \mathsf{cl}(\mathsf{E}(\Phi)).$$

We get similar properties:

Lemma 45 $\mathsf{I}(\Phi) \subseteq \mathsf{SN}$, $\mathsf{E}(\Phi) \cap \mathsf{SN} \in \mathsf{SAT}$, *and* $\mathsf{I}(\Phi) \subseteq \mathsf{E}(\Phi)$.

Proof

(1) Let $r \in \mathsf{S}_\alpha(\Phi)$. Then for $\sigma := \alpha$ and $\mathcal{M} := \mathsf{SN}$, we get $r = r[\alpha := \sigma] \in \Phi(\mathcal{M}) \subseteq \mathsf{SN}$, hence also $\Lambda\alpha r \in \mathsf{SN}$.

(2) Check the conditions of saturatedness for $\mathsf{E}(\Phi) \cap \mathsf{SN}$:

1. Trivial.
2. Let the terms among $\vec{R}$ be in SN, $\sigma \in \mathcal{T}_u$ and $\mathcal{M} \in \mathsf{SAT}$. Since $\Phi(\mathcal{M}) \in \mathsf{SAT}$, $x\vec{R}\sigma \in \Phi(\mathcal{M})$.

3./4. Append σ and use saturatedness of $\Phi(\mathcal{M})$.

(3) Let $r \in \mathsf{S}_\alpha(\Phi)$, $\sigma \in \mathcal{T}_u$ and $\mathcal{M} \in \mathsf{SAT}$. Show that $(\Lambda\alpha r)\sigma \in \Phi(\mathcal{M})$. Since $\Phi(\mathcal{M}) \in \mathsf{SAT}$, if suffices to show $r[\alpha := \sigma] \in \Phi(\mathcal{M})$ which follows by definition of $\mathsf{S}_\alpha(\Phi)$. □

From the lemma, we get $\mathsf{I}(\Phi) \subseteq \forall_{\mathrm{I}}\Phi$, $\forall_{\mathrm{E}}\Phi = \mathsf{E}(\Phi) \cap \mathsf{SN}$ and $\forall_{\mathrm{I}}\Phi \subseteq \forall_{\mathrm{E}}\Phi$. Define $\forall\Phi := \forall_{\mathrm{X}}\Phi$ with $\mathrm{X} \in \{\mathrm{I}, \mathrm{E}\}$. As usual, we never use any property depending on this choice but only:

(SAT) $\forall\Phi \in \mathsf{SAT}$.

$(\forall_{\mathrm{I}})$ If $r \in \mathsf{S}_\alpha(\Phi)$ then $\Lambda\alpha r \in \forall\Phi$.

$(\forall_{\mathrm{E}})$ $r \in \forall\Phi \wedge \sigma \in \mathcal{T}_u \wedge \mathcal{M} \in \mathsf{SAT} \Rightarrow r\sigma \in \Phi(\mathcal{M})$.

The notion of candidate assignment remains unchanged:

Definition 35 (Candidate assignment) *Any finite set of pairs (written $\alpha : \mathcal{M}$), consisting of a type variable and a saturated set, such that no type variable occurs twice.*

Definition 36 (Strong computability) *Define the saturated set* $\mathsf{SC}^\rho[\Gamma]$ *of* strongly computable terms *w. r. t. type* ρ *and the candidate assigment* Γ *by recursion on* ρ*:*

- $\mathsf{SC}^{\alpha}[\Gamma] := \begin{cases} \mathcal{M} & \textit{if } \alpha : \mathcal{M} \in \Gamma, \\ \mathsf{SN} & \textit{otherwise.} \end{cases}$
- $\mathsf{SC}^{\rho \to \sigma}[\Gamma] := \mathsf{SC}^{\rho}[\Gamma] \to \mathsf{SC}^{\sigma}[\Gamma]$.
- *Define* $\Phi : \mathsf{SAT} \to \mathsf{SAT}$ *by setting* $\Phi(\mathcal{M}) := \mathsf{SC}^{\rho}[\Gamma, \alpha : \mathcal{M}]$. *Set* $\mathsf{SC}^{\forall \alpha \rho}[\Gamma] := \forall \Phi$.

Lemma 46 (Coincidence) *If* $\alpha \notin FV(\rho)$ *then* $\mathsf{SC}^{\rho}[\Gamma, \alpha : \mathcal{M}] = \mathsf{SC}^{\rho}[\Gamma]$.

Proof Induction on ρ. □

Lemma 47 (Substitution) $\mathsf{SC}^{\rho[\alpha:=\sigma]}[\Gamma] = \mathsf{SC}^{\rho}[\Gamma, \alpha : \mathsf{SC}^{\sigma}[\Gamma]]$.

Proof Induction on ρ, using the previous lemma. □

As in Lemma 34, we make use of the simultaneous substitution $r[\vec{x} := \vec{s}]$ of all occurrences of x_i by s_i (for every i, with different variables x_i) in r and moreover of the simultaneous substitution $r[\vec{\alpha} := \vec{\sigma}]$ of all occurrences of α_i by σ_i (for every i, with different variables α_i) in r which both may be defined by recursion on r like the ordinary notions of substitution. We will again use the notation $\vec{x} : \vec{\rho}$ for $x_1 : \rho_1, \ldots, x_n : \rho_n$, $\vec{s} \in \mathsf{SC}^{\vec{\rho}}[\Gamma]$ for $s_1 \in \mathsf{SC}^{\rho_1}[\Gamma], \ldots, s_n \in \mathsf{SC}^{\rho_n}[\Gamma]$, and moreover, $\vec{\alpha} : \vec{\mathcal{M}}$ for $\alpha_1 : \mathcal{M}_1, \ldots, \alpha_m : \mathcal{M}_m$.

Lemma 48 *If* $\vec{x} : \vec{\rho} \vdash r : \rho$, $\vec{s} \in \mathsf{SC}^{\vec{\rho}}[\vec{\alpha} : \vec{\mathcal{M}}]$ *and* $\vec{\sigma}$ *is a list of types corresponding to* $\vec{\alpha}$ *then* $r[\vec{\alpha} := \vec{\sigma}][\vec{x} := \vec{s}] \in \mathsf{SC}^{\rho}[\vec{\alpha} : \vec{\mathcal{M}}]$.

Proof By induction on $\vec{x} : \vec{\rho} \vdash r : \rho$ simultaneously for every candidate assignment.

(V) $r : \rho = x_i : \rho_i$. Obvious.

($\to_I$) Let $\vec{x} : \vec{\rho} \vdash \lambda x^{\rho} r : \rho \to \sigma$ thanks to $\vec{x} : \vec{\rho}, x : \rho \vdash r : \sigma$. We have to show that $(\lambda x^{\rho} r)[\vec{\alpha} := \vec{\sigma}][\vec{x} := \vec{s}] \in \mathsf{SC}^{\rho}[\vec{\alpha} : \vec{\mathcal{M}}] \to \mathsf{SC}^{\sigma}[\vec{\alpha} : \vec{\mathcal{M}}]$. We may assume that $x \notin \vec{x} \cup \mathrm{FV}(\vec{s})$, and hence

$$(\lambda x^{\rho} r)[\vec{\alpha} := \vec{\sigma}][\vec{x} := \vec{s}] = \lambda x^{\rho[\vec{\alpha}:=\vec{\sigma}]}.r[\vec{\alpha} := \vec{\sigma}][\vec{x} := \vec{s}].$$

It suffices to show that $r[\vec{\alpha} := \vec{\sigma}][\vec{x} := \vec{s}] \in \mathsf{S}_x(\mathsf{SC}^{\rho}[\vec{\alpha} : \vec{\mathcal{M}}], \mathsf{SC}^{\sigma}[\vec{\alpha} : \vec{\mathcal{M}}])$. So assume $s \in \mathsf{SC}^{\rho}[\vec{\alpha} : \vec{\mathcal{M}}]$ and show $r[\vec{\alpha} := \vec{\sigma}][\vec{x} := \vec{s}][x := s] \in \mathsf{SC}^{\sigma}[\vec{\alpha} : \vec{\mathcal{M}}]$. This follows from the induction hypothesis since, by our assumption,

$$r[\vec{\alpha} := \vec{\sigma}][\vec{x} := \vec{s}][x := s] = r[\vec{\alpha} := \vec{\sigma}][\vec{x}, x := \vec{s}, s].$$

($\to_E$) This is an immediate consequence of the induction hypothesis and the rule ($\to_E$) for saturated sets.

($\forall_I$) Let $\vec{x} : \vec{\rho} \vdash \Lambda \alpha r : \forall \alpha \rho$ thanks to $\vec{x} : \vec{\rho} \vdash r : \rho$ and $\alpha \notin \mathrm{FV}(\vec{\rho})$. We may assume that $\alpha \notin \vec{\alpha} \cup \mathrm{FV}(\vec{\sigma}) \cup \mathrm{FTV}(\vec{s})$. Therefore, $(\Lambda \alpha r)[\vec{\alpha} := \vec{\sigma}][\vec{x} := \vec{s}] = \Lambda \alpha.r[\vec{\alpha} := \vec{\sigma}][\vec{x} := \vec{s}]$. Let $\sigma \in \mathcal{T}_u$ and $\mathcal{M} \in \mathsf{SAT}$. We have to show that $r[\vec{\alpha} := \vec{\sigma}][\vec{x} := \vec{s}][\alpha := \sigma] \in \mathsf{SC}^{\rho}[\vec{\alpha} : \vec{\mathcal{M}}, \alpha : \mathcal{M}]$. Since $\alpha \notin \mathrm{FV}(\vec{\rho})$, we may apply the Coincidence Lemma and get $\vec{s} \in \mathsf{SC}^{\vec{\rho}}[\vec{\alpha} : \vec{\mathcal{M}}, \alpha : \mathcal{M}]$. We are done by the induction hypothesis since

$$r[\vec{\alpha} := \vec{\sigma}][\vec{x} := \vec{s}][\alpha := \sigma] = r[\vec{\alpha}, \alpha := \vec{\sigma}, \sigma][\vec{x} := \vec{s}].$$

($\forall_E$) Let $\vec{x} : \vec{\rho} \vdash r\sigma : \rho[\alpha := \sigma]$ thanks to $\vec{x} : \vec{\rho} \vdash r : \forall \alpha \rho$. By induction hypothesis, $r[\vec{\alpha} := \vec{\sigma}][\vec{x} := \vec{s}] \in \mathsf{SC}^{\forall \alpha \rho}[\vec{\alpha} : \vec{\mathcal{M}}]$. Set $\mathcal{M} := \mathsf{SC}^{\sigma}[\vec{\alpha} : \vec{\mathcal{M}}]$. Then $r[\vec{\alpha} := \vec{\sigma}][\vec{x} := \vec{s}]\sigma[\vec{\alpha} := \vec{\sigma}] \in \mathsf{SC}^{\rho}[\vec{\alpha} : \vec{\mathcal{M}}, \alpha : \mathcal{M}] = \mathsf{SC}^{\rho[\alpha:=\sigma]}[\vec{\alpha} : \vec{\mathcal{M}}]$ by the Substitution Lemma. □

By setting $s_i := x_i \in \mathsf{SC}^{\rho_i}[\emptyset]$ for $\Gamma = \vec{x} : \vec{\rho}$, and by using $\mathsf{SC}^{\rho}[\emptyset] \subseteq \mathsf{SN}$, we get the following

Theorem 5 (Strong normalization of F) *If* $\Gamma \vdash r : \rho$ *then* $r \in \mathsf{SN}$. □

As an application, we prove:

Lemma 49 (Typed confluence) *If* $\Gamma \vdash r : \rho$, $r \to^*_{\beta\eta} r'$ *and* $r \to^*_{\beta\eta} r''$ *then there is a term* t *such that* $r' \to_{\beta\eta} t$ *and* $r'' \to_{\beta\eta} t$.

Proof Induction on $r \in \mathsf{sn}$.[14] If $r = r'$ or $r = r''$ then the claim is trivial (set $t = r''$ or $t = r'$, respectively). Otherwise, $r \to_{\beta\eta} r'_0 \to^*_{\beta\eta} r'$ and $r \to_{\beta\eta} r''_0 \to^*_{\beta\eta} r''$ for some terms r'_0 and r''_0. By typed local confluence, there is a term s such that $r'_0 \to^*_{\beta\eta} s$ and $r''_0 \to^*_{\beta\eta} s$. By Subject Reduction, $\Gamma \vdash r'_0 : \rho$ and $\Gamma \vdash r''_0 : \rho$. Hence, by induction hypothesis for r'_0, there is s' such that $r' \to^*_{\beta\eta} s'$ and $s \to^*_{\beta\eta} s'$, hence $r''_0 \to^*_{\beta\eta} s'$. We now apply the induction hypothesis to r''_0 and get the term t with and $r'' \to^*_{\beta\eta} t$ and $s' \to^*_{\beta\eta} t$, hence also $r' \to^*_{\beta\eta} t$. □

8 Monotone Inductive Types

This section is based on [Mat99].

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[14] This proof is in essence the proof of Newman's Lemma saying that a locally confluent and strongly normalizing binary relation is confluent.