Corollary 25 $\Gamma \not\vdash \Omega : \rho$.

Proof If $\Gamma \vdash \Omega : \rho$ then also for some $\rho \in \mathcal{P}$. By Inversion, there is $\vec{\rho}$ and σ such that $\rho \in \vec{\rho}$ and $\Gamma \vdash \omega : \sigma \to \cap\vec{\rho}$ and $\Gamma \vdash \omega : \sigma$. Since $\sigma = \cap\vec{\sigma}$ with $\vec{\sigma} \subset \mathcal{P}$, the previous lemma applies. $\square$

6.3 Strong Normalization of Typable Terms

Lemma 26 *If* $r \in \mathsf{SN}$ *and* $x \notin FV(r)$ *then* $rx \in \mathsf{SN}$.

Proof Induction on SN. $\square$

Note that $x \notin \mathrm{FV}(r)$ is a superfluous assumption. Nevertheless, we do not need a stronger statement, and therefore only consider what can be proved so easily.

Lemma 27 *If* $r, s \in \mathsf{SN}$ *and* $\Gamma \vdash s : \rho$ *and* $\Gamma, x : \rho \vdash r : \sigma$ *then* $r[x := s] \in \mathsf{SN}$.

Proof We first define a measure $\mathsf{h}(\rho) \in \mathbb{N}_0$ for every type ρ by recursion on ρ as follows:

$$\begin{aligned} \mathsf{h}(\iota) &:= 0 \\ \mathsf{h}(\rho \to \sigma) &:= 1 + \max(\mathsf{h}(\rho), \mathsf{h}(\sigma)) \\ \mathsf{h}(\rho \cap \sigma) &:= \max(\mathsf{h}(\rho), \mathsf{h}(\sigma)) \end{aligned}$$

The proof is by main induction on $\mathsf{h}(\rho)$, side induction on $r \in \mathsf{SN}$ and case distinction according to $r \in \mathsf{SN}$. (Observe that we also have $\Gamma \vdash r[x := s] : \sigma$ by Lemma 21(a).)

Case $y\vec{r}$. This has been derived from $\vec{r} \subset \mathsf{SN}$. By multiple Inversion, we get $\Gamma, x : \rho \vdash \vec{r} : \vec{\mu}$ for suitable $\vec{\mu}$. By side induction hypothesis, $\vec{r}[x := s] \subset \mathsf{SN}$. Therefore, $y\vec{r}[x := s] \in \mathsf{SN}$. This finishes with the case $y \neq x$. So assume $y = x$. If $\vec{r}$ is empty, the claim is trivial. Otherwise, $\vec{r} = t, \vec{t}$, hence $\Gamma, x : \rho \vdash xt\vec{t} : \sigma$. $xt\vec{t} = (z\vec{t})[z := xt]$ for some "new" variable z. By Lemma 21(b), there is a type τ such that $\Gamma, x : \rho, z : \tau \vdash z\vec{t} : \sigma$ and $\Gamma, x : \rho \vdash xt : \tau$. $\tau = \cap\vec{\tau}$. Consider the element τ_k of $\vec{\tau}$. $\Gamma, x : \rho \vdash xt : \tau_k$. By Inversion, there is a type ρ'_k and types $\vec{\nu}_k$ with $\tau_k \in \vec{\nu}_k$ such that $\Gamma, x : \rho \vdash x : \rho'_k \to \cap\vec{\nu}_k$ and $\Gamma, x : \rho \vdash t : \rho'_k$. With $\rho = \cap\vec{\rho}$ and Inversion, we get $\rho'_k \to \cap\vec{\nu}_k \in \vec{\rho}$. Therefore, $\mathsf{h}(\tau_k) \leq \mathsf{h}(\cap\vec{\nu}_k) < \mathsf{h}(\rho'_k \to \cap\vec{\nu}_k) \leq \mathsf{h}(\rho)$. Since this holds for every k, also $\mathsf{h}(\tau) < \mathsf{h}(\rho)$. We continue with $k := 1$, $\rho' := \rho'_1$ and $\vec{\nu} := \vec{\nu}_1$. Clearly, $\Gamma \vdash s : \rho' \to \cap\vec{\nu}$. By the previous lemma, $sz' \in \mathsf{SN}$ with a "new" variable z'. By Weakening 1, $\Gamma, z' : \rho' \vdash sz' : \cap\vec{\nu}$. By Lemma 21(a): $\Gamma \vdash t[x := s] : \rho'$. Since $\mathsf{h}(\rho') < \mathsf{h}(\rho' \to \cap\vec{\nu}) \leq \mathsf{h}(\rho)$ we may apply the main induction hypothesis for type ρ' and get $st[x := s] \in \mathsf{SN}$. By Lemma 21(a), $\Gamma, z : \tau \vdash z\vec{t}[x := s] : \sigma$ and $\Gamma \vdash st[x := s] : \tau$. Clearly, $z\vec{t}[x := s] \in \mathsf{SN}$. Hence, by the main induction hypothesis for type τ, we finally get $(x\vec{r})[x := s] = st[x := s]\vec{t}[x := s] \in \mathsf{SN}$.

Case λyr. This comes from $r \in \mathsf{SN}$. We have $\Gamma, x : \rho \vdash \lambda yr : \sigma$. Show $(\lambda yr)[x := s] \in \mathsf{SN}$. We may assume that $y \notin \{x\} \cup \mathrm{FV}(s)$. Therefore $(\lambda yr)[x := s] = \lambda y.r[x := s]$, and it suffices to show $r[x := s] \in \mathsf{SN}$. $\sigma = \cap\vec{\sigma}$

and $\Gamma, x : \rho \vdash \lambda y r : \sigma_1$ (the first element of $\vec{\sigma}$). By Inversion, $\sigma_1 = \tau \to \sigma'$ and $\Gamma, x : \rho, y : \tau \vdash r : \sigma'$. By Weakening 1, $\Gamma, y : \tau \vdash s : \rho$. The side induction hypothesis yields $r[x := s] \in \mathsf{SN}$.

Case $(\lambda y r) s \vec{s}$. This is derived from $r[y := t]\vec{t} \in \mathsf{SN}$ and $t \in \mathsf{SN}$. We have $\Gamma, x : \rho \vdash (\lambda y r) t \vec{t} : \sigma$ and have to show that (we assume that $y \notin \{x\} \cup FV(s)$) $((\lambda y r) t \vec{t})[x := s] = (\lambda y . r[x := s]) t[x := s] \vec{t}[x := s] \in \mathsf{SN}$. For this we need $(r[x := s])[y := t[x := s]]\vec{t}[x := s] \in \mathsf{SN}$ and $t[x := s] \in \mathsf{SN}$. By multiple Inversion, we get $\Gamma, x : \rho \vdash t : \mu$ for some type μ. The side induction hypothesis yields $t[x := s] \in \mathsf{SN}$. By Subject Reduction, $\Gamma, x : \rho \vdash r[y := t]\vec{t} : \sigma$. Hence, again by side induction hypothesis, $(r[y := t]\vec{t})[x := s] \in \mathsf{SN}$. We are done by Lemma 2 which tells us that $(r[y := t]\vec{t})[x := s] = (r[x := s])[y := t[x := s]]\vec{t}[x := s]$. $\square$

Corollary 28 (Main Theorem) *If* $\Gamma \vdash r : \rho$ *then* $r \in \mathsf{SN}$.

Proof By induction on $\vdash$. In the case $(\to_E)$ use the induction hypothesis, $rs = (rx)[x := s]$ and the preceding two lemmas. $\square$

Since SN is the set of strongly normalizing terms (here we only need soundness of SN), every typable term is strongly normalizing. Since typability with simple types is more restrictive than that with intersection types, simple typability also implies strong normalization.

6.4 Typability of Strongly Normalizing Terms

Let us write $\Gamma + \Delta$ for the context where the requirements on the variables in Γ and Δ are added. More formally, if $\Gamma = \vec{x} : \vec{\rho}, \vec{y} : \vec{\sigma}$ and $\Delta = \vec{x} : \vec{\rho}\,', \vec{z} : \vec{\tau}$ with $\vec{x}$, $\vec{y}$ and $\vec{z}$ pairwise disjoint and $\vec{x}$ of length n, then $\Gamma + \Delta := x_1 : \rho_1 \cap \rho_1', \dots, x_n : \rho_n \cap \rho_n', \vec{y} : \vec{\sigma}, \vec{z} : \vec{\tau}$. From Weakening 1 and Weakening 2, it follows that $\Gamma \vdash r : \rho$ implies $\Gamma + \Delta \vdash r : \rho$. We will also form $\Gamma_1 + \dots + \Gamma_n$ in a similar way without caring about parentheses.

Theorem 2 (Completeness) *If* $r \in \mathsf{SN}$ *then there are* Γ *and* ρ *such that* $\Gamma \vdash r : \rho$.

Proof Induction on $r \in \mathsf{SN}$.

$x\vec{r}$. By induction hypothesis, $\Gamma_i \vdash r_i : \rho_i$ for every i. Therefore

$$\Gamma_1 + \dots + \Gamma_n + x : (\rho_1 \to \dots \to \rho_n \to \rho) \vdash x\vec{r} : \rho$$

for any type ρ.

$\lambda x r$. Let $\lambda x r \in \mathsf{SN}$ due to $r \in \mathsf{SN}$. By induction hypothesis, $\Gamma \vdash r : \rho$. Possibly by Weakening 1, we may assume that $\Gamma = \hat{\Gamma}, x : \sigma$. This yields $\hat{\Gamma} \vdash \lambda x r : \sigma \to \rho$.

$(\lambda x r)s\vec{s}$. Assume $(\lambda x r)s\vec{s} \in \mathsf{SN}$ has been derived from $r[x := s]\vec{s} \in \mathsf{SN}$ and $s \in \mathsf{SN}$. By induction hypothesis, $\Gamma \vdash r[x := s]\vec{s} : \tau$ and $\Gamma' \vdash s : \sigma$. Setting $\Delta := \Gamma + \Gamma'$, we get $\Delta \vdash r[x := s]\vec{s} : \tau$ and $\Delta \vdash s : \sigma$. Writing $r[x := s]\vec{s} = (y\vec{s})[y := r[x := s]]$ and applying Lemma 21(b), we find a type ρ such that $\Delta \vdash y\vec{s} : \tau$ and $\Delta \vdash r[x := s] : \rho$. Again by Lemma 21(b), there is a type σ' such that $\Delta, x : \sigma' \vdash r : \rho$ and $\Delta \vdash s : \sigma'$. Hence, $\Delta \vdash (\lambda x r)s : \rho$ and by Lemma 21(a), $\Delta \vdash (\lambda x r)s\vec{s} : \tau$. $\square$

7 Monotone Inductive Types

This section is based on [Mat99].

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